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H_infinity Filtering and Control for Networked Systems: Stochastic Parameterized Method

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Preface

Networked Control System (NCS) refers to a class of spatially distributed closed-loop systems where the actuators, sensors and controllers are connected by communication networks. Compared with the traditional control system, NCS has the advantages of simple connection, easy expansion, convenient for maintenance and strong interaction, *etc*. It is widely applied in many fields of national economy, such as remote medical treatment, intelligent transportation, aerospace, national defense and so on. But due to the limited network bandwidths, there exists problems inevitable such as packet losses and varying delays *etc*. during the transmission of data packets. At the same time, the introduction of the networks increases the complexity of the systems such that the systems will suffer from nonlinear disturbances caused by the environment and the sensor saturations. Due to the random nature of data arrival, the whole control system is no longer a deterministic system, but a stochastic system. At present, the research on network-based stochastic system has become one of the international hot topics in the control field. In this book, we investigate the H_∞ filtering and control problems based on stochastic parameterized method for the NCS in the consideration of all kinds of uncertainties during the data transmission. By using Lyapunov stability theory and linear matrix inequality technique combining with stochastic analysis and robust control theory, the desired H_∞ filter or controller is designed such that the filtering error system or the closed-loop controlled system is stable in the mean square sense to some extent. Simulation results are also given to show the effectiveness of the proposed method.

This book contains eight chapters, which is organized as follows. After an introduction to research background and significance, the development history, the uncertain factors and the main research status of the NCS are addressed in Chapter 1. The following seven chapters are divided into two parts. One is about filtering problem, which includes from Chapter 2 to Chapter 5. The other is about control problem, which includes from Chapter 6 to Chapter 8. In Chapter 2, H_∞ filtering is investigated for linear systems by considering multiple packet dropouts and random delays in network transmission. In Chapter 3, H_∞ filtering design problem is considered for multiple-channel NCS with not only consecutive packet losses and varying delays but also randomly occurred nonlinearities. In Chapter 4, H_∞ filtering is studied for NCS with delayed measurements, packet losses and randomly varying nonlinearities in a multiple packets compensation strategy. In Chapter 5, H_∞ filtering is dealt with for fuzzy-model-based nonlinear NCS subject to sensor saturations. In Chapter 6, H_∞ control problem is considered for NCS with one-step random delays and packet dropouts. In Chapter 7, H_∞ control is designed based on an observer for NCS with consecutive packet dropouts and multiple packet dropouts. In Chapter 8, H_∞ control is investigated where the NCS is stochastic nonlinear and the randomly occurred sensor saturations, multiple delays and packet dorpouts are considered. At the end of each chapter, the MATLAB simulation results are given for demonstrating the effectiveness of the proposed method. Moreover, each chapter is interrelated and independent. The author can read this book according to your own needs without front to back.

The publication of this book is supported by National Natural Science Foundation of China under grant NSFC-61503126, Natural Science Foundation of Heilongjiang Province under grant F2018024, Basic Research Fund of Heilongjiang University in Heilongjiang Province under grant RCYJTD201806 and Shanghai Key Laboratory of Multidimensional Information Processing, East China Normal University under grant 2019MIP003. I would like to express my heartfelt thanks for the support. I have been engaged in the research of NCS since I studied for my Ph.D in Heilongjiang University. This book is mainly the achievements of mine during the doctoral period. I have been guided and taught by many predecessors and benefited a lot. Thanks for help from the predecessors and colleagues in NCS. I have been working in Heilongjiang University since 2005. I sincerely expressed my gratitude for my supervisors Prof. Zhigang Han and Prof. Shuli Sun. Their rigorous style of study and professionalism have a profound impact on me. I transfer to Shanghai Institute of Technology last year. Along the way, there are many people who care about me, support me and encourage me. This book is dedicated to them.

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Notation

Symbol	Description
$\mathbb{R}^n$	n-dimensional real Euclidean space
$\mathbb{R}^{n\times m}$	Set of $n \times m$ real matrix
I	Identity matrix with suitable dimension
0	Zero matrix with suitable dimension
X^T	Transpose of matrix X
X^{-1}	Inverse of matrix X
$X^\dagger$	Moore-Penrose inverse of matrix X
$X > 0$	Symmetric positive definite matrix X
$X \geq 0$	Symmetric positive semi-definite matrix X
$X > Y$	Symmetric positive definite matrix $X - Y$
$X \geq Y$	Symmetric positive semi-definite matrix $X - Y$
$diag\{A_1, \cdots, A_n\}$ or $diag_n\{A_j\}$ $(j = 1, \cdots, n\}$	Block diagonal matrix with $A_j(j = 1, \cdots, n)$ on the diagonal
$vec_n\{A_j\}$ $(j = 1, \cdots, n\}$	Vector with elements $A_j(j = 1, \cdots, n)$
$\otimes$	Kronecker product
$*$	Symmetric terms in a symmetric matrix
$prob\{\cdot\}$	Occurrence probability of event "·"
$E\{x\}$	Expectation of stochastic variable x
$\lambda_{max}\{X\}$	Maximum singular value of matrix X
$\lambda_{min}\{X\}$	Minimum singular value of matrix X
I^+	Set of positive integers
$l_2[0, +\infty)$	Square summable vector space
$\|\cdot\|$	Vector or matrix norm

Chapter 1 Introduction

1.1 Research background and significance

With the rapid development of science and technology, the controlled objects faced by people are more and more complex. Controlled objects, sensors and controllers exist across regions, and the overall controlled systems exhibit distributed and complicated [1]. The traditional centralized control and distributed control of such systems cannot meet the increasing performance requirements. The networked control system (NCS) emerged with the booming computer technology, network communication technology and control technology provides an effective way to solve such problems. NCS refers to the sensors, controllers and actuators distributed in different geographical locations are connected through the network, which forms a fully distributed real-time feedback closed-loop control system. It has become one of the research hotspots in the field of control worldwide [2,3].

The outstanding advantage of NCS is the reduced complexity of the design and implementation of large-scale control system. Compared with the traditional point-to-point control system, NCS has the shared information, less system wires, higher system flexibility and reliability, and is easy to expand and maintain. Thus, it has been widely used in various fields of the national economy. For example, in the field of industrial control, the fieldbus-based NCS has been successfully used in Siemens and Rockwell; in the field of transportation, the intelligent transportation system based on NCS develops rapidly, and attracts much attention; in the medical field, remote surgery using NCS has become a reality; in the military field, the large-scale offense and defense system based on NCS is an effective system through the cooperative control of the weapon platform [4]. In addition, wireless NCS can be applied in high-risk environments, which cannot be achieved by the traditional control system.

The introduction of the network brings not only the great convenience for the industrial control application, but also great challenges for the analysis and synthesis of the control system. The shared communication network in the system has caused a fundamental change in the information transmission mode. Due to the limitation of network bandwidth, bearing capacity and service capability, the data packets inevitably occur some network-induced problems during the transmission, such as the delay, loss, bit error and timing disorder, *etc*. [5,6]. Therefore, in the analysis and design of the NCS, we must consider the unique characteristics of the data transmission through the network. For example [7],

(1) The loss of regularity. The time that the data arrives is not regular, which can no longer be characterized by simple sampling time.

(2) The loss of integrity. Due to the loss and error of data in the transmission, the data is incomplete. There may be similar problems in digital control system, but the possibility of occurrence is at different magnitudes;

(3) The loss of causality. Due to the time uncertainty of network transmission, early generated data can arrive at the remote-control system later than the late generated data. Therefore, the order of data arrival no longer obeys the causality.

(4) The loss of certainty. Due to the randomness of data arrival, the entire control process is a stochastic system rather than a deterministic system, which is the most important theoretical feature based on networked control.

NCS is a stochastic system in nature. In NCS, various uncertain factors affect the performance of the system, such as the random delay with uncertain distribution law, the uncertain rate of packet losses, uncertain sampling rates, random measurement noise and process noise, the externally randomly occurring nonlinear disturbance, externally random cyber attack, *etc*. When these issues exist, the design of controller and filter with certain interference suppression performance and ensuring the mean-square stability of closed-loop system is a major research direction of NCS. The robust control is the application of the uncertainty principle in disciplines of system and control. The basic feature is to use a controller with fixed structure and parameters to ensure that the design requirement can be met even when the uncertainty has the most pernicious effect on the system performance. In the early days, the method of algebraic Riccati equation had been widely used. It transforms the robust analysis and synthesis problem of the system into a Riccati-type matrix equation, then the robust performance condition and the design method of robust controller can be obtained by solving the Riccati equation. Now, the method based on linear matrix inequality (LMI) is mainly used. The advantage of this method is that the controller design can be obtained by relatively direct matrix operations, and there are no excessive restrictions on the system model. At present, LMI has become one of the mainstream technologies for dealing with the robust performance analysis and synthesis problem, which greatly promotes the development of robust control theory.

After all, we are in a world of movement and change. Thus, uncertainty is inevitable and a permanent obstacle to achieving omniscience. The profound philosophical implication of uncertainty, as well as the guidance of all aspects of human activities, are increasingly highlighting its charm. Mr. Yu-Chi Ho has a saying in “Blog Articles on Sciences and Life”: “It is precisely because of this uncertainty that your actual life cannot be predicted, the world will be colorful, and you will feel that life is meaningful.” Research on NCS can not only provide new ideas and methods in engineering practice, but also theoretically promote the cross-infiltration and integration of multidisciplinary technologies, such as automatic control technology, computer technology and communication technology. Then, it makes control technology enter a new stage of development.

1.2 Development of NCS

NCS is a fully distributed real-time feedback control system that connects sensors, actuators and controllers distributed in different geographical locations through the network to form a closed loop. Control technology, computer technology and network communication technology are the technical basis and important driving forces for the generation and development of NCS. In addition, they are key factors that determine the continuous improvement and innovation of NCS.

The development of NCS can be roughly divided into three stages [8,9].

(1) Distributed Control System (DCS).

Prior to the advent of DCS, the computer control system was Direct Digital Control (DDC), in which all sensors and actuators were connected point-to-point with the same computer. Because the computer was expensive at the time, the system generally adopted a centralized architecture. The entire production process and control strategy were completed by one computer. Then, even a single malfunction of the computer would invalidate the entire system and all the circuits.

With the rapid development of electronic technology, DCS appeared in the 1970s. The basic idea is that DCS adopts “centralized management, decentralized control”, and then the damage due to the unreliability of certain part to the entire system is reduced to a low degree. The DCS consists of three levels: operator station, field control station and engineer station. A typical DCS is shown in Figure 1.1.

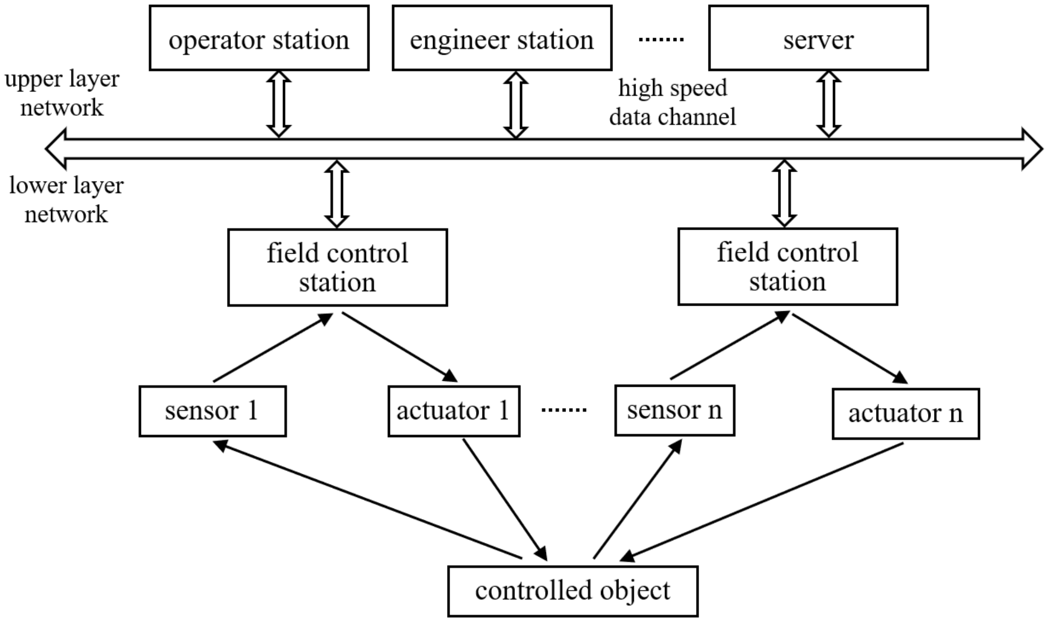

Figure 1.1 Structure diagram of typical DCS.

The DCS distributes the control tasks to several small computer controllers (field control stations), each of which adopts a DDC architecture to conduct part of the control loop. The computer control network is established between the controller and controller, as well as the controller and the host computer (operator station or the engineer station). The operator in this control architecture can monitor the real-time running status of the controlled system in the host computer. In addition, the control strategy of a control loop can be configured in the host computer and be downloaded to the corresponding controller by controlling network to run in real time.

DCS greatly improves the reliability of the control system, and realizes centralized management and decentralized control. It has a certain networked content. However, it is limited by the level of computer and network technology at that time, and has obvious disadvantages. First, the DCS architecture is a multi-level master-slave relationship, and the communication among lower layer computers must pass through the host computer. Then, the host computer is overloaded, and the efficiency is low. Once the host computer malfunctions, the entire system will “hook”. Second, DCS is a digital-analog hybrid system. The field instrument still adopts the traditional 4~20 mA analog signal, which has high engineering and management cost as well as poor flexibility. Third, standards of DCS are not uniform, and systems are incompatible with each other. This is not conducive to further improving the flexibility and maintainability of the system configuration.

(2) Fieldbus Control System (FCS)

The FCS produced in the 1980s is a new generation control system after the DCS. It is a real-time control system, and uses the open fieldbus as the communication network, by which field controllers and smart instruments are interconnected and communicated. In addition, it is a fully digital, fully decentralized, interoperable and open control network.

FCS overcomes shortcomings of DCS. First, FCS extends the control network to the control equipment at the production site, and the signal transmission is completely digitized, which can improve the accuracy and reliability of signal conversion. Second, because the smart instrument (transmitter, actuator) of FCS is equipped with a microprocessor, it can directly form a control loop at the production site, and the control function can be fully decentralized. The DCS architecture is centralized and distributed, and the FCS architecture is fully distributed. The control function has been completely decentralized to the site, and achieved completely decentralized control. Third, it breaks through the limitation of DCS using the private communication network, and adopts a scheme based on openness and standardization to overcome defects caused by the closed system. The structure of a typical FCS is shown in Figure 1.2.

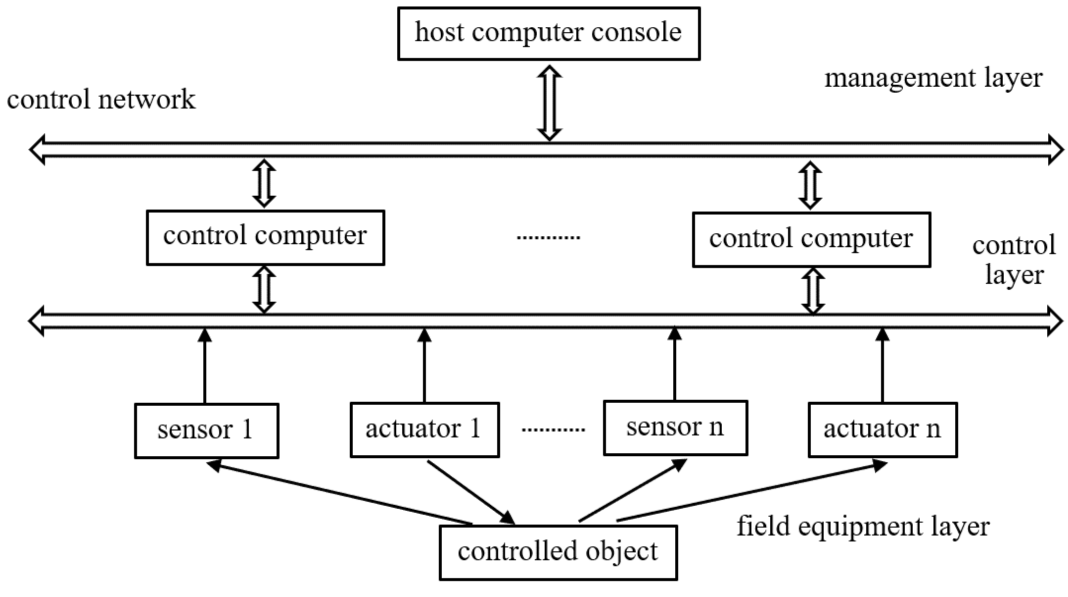

Figure 1.2 Structure diagram of typical FCS.

After more than 20 years of development, FCS technology has achieved great success, and has been widely used in many fields. However, there are some problems that limit its further application. First, the fieldbus standards are not uniform. Although the International Electrotechnical Commission (IEC) has reached the international bus standard, there are more than 10 types of buses. Each manufacturer is self-contained, and cannot be completely open. The interchange and interoperating are difficult to achieve. Second, the fieldbus is a layered private network. The upper management information system of the fieldbus mostly adopts the TCP/IP protocol. The upper layer protocol is incompatible with the lower one. This results in the separation of management and control, and it is difficult to realize the comprehensive automation and remote control of the entire factory.

(3) Networked Control System (NCS)

With the development of microprocessor and communication technology, the emergence of processor with higher performance and network with higher bandwidth has made it possible to apply TCP/IP protocols to real-time measurement and control systems. This provides an information platform for building NCS, and results in extensive research on public network-based control system that emerged in the 1990s. It is a completely networked and distributed control system. This open NCS based on public network has outstanding advantages and features. It can use on-site intelligent equipment to complete the control task and realize network communication, as well as use network technology to form the operating platform of the control system. This achieves the seamless integration of the networked control and information network, as well as the integration of management and control, which results in a high-performance control system. The structure of a typical NCS is shown in Figure 1.3.

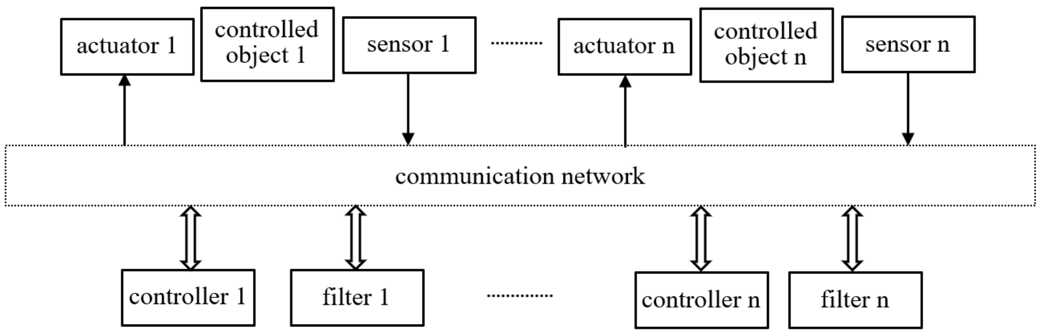

Figure 1.3 Structure diagram of typical NCS.

Narrow-sense NCS refers to the closed-loop control system formed by the communication network replacing the traditional point-to-point line between the sensor, actuator and controller. In a broader sense, the NCS includes the remote control, information transmission and optimization can be realized between the factory workshop, production line and engineering field equipment through the internet and the enterprise information network. In NCS, as long as a client software is installed, a control engineer with access can monitor the control loop of a control network on any computer connected to the internet around the world without returning to the site. This greatly improves the work efficiency. In recent years, the trend of integrated equipment has evolved from the wired communication to wireless communication. By adopting wireless network technologies, such as wireless sensors, it is possible to implement NCS with the special application. The wireless network has an irreplaceable role when the object is movable or the environment in which the location of the object is difficult to connect with the wired network.

At present, the declining price of network hardware and control equipment, and the flourishing software industry extend the application of NCS to all areas of the national economy. Establishing and perfecting systematic networked control theory and method is an urgent and important issue, which is also the demand of the times, and is of great significance to the application and maintenance of NCS.

1.3 Network-induced uncertainties of NCS

NCS is a complex control system integrating communication network and control system. The unreliable network environment makes the whole NCS an uncertain stochastic system. The uncertain factors induced by the network environment can be divided into two categories. One is the uncertainty of data packets in the network transmission process due to the limited network bandwidth resources, such as the time delay, packet loss, timing disorder, multi-packet transmission, *etc*. The other is the factor due to external environment, such as the multi-rate sampling, unpredictable nonlinear interference input or cyber attack. These issues will not only reduce the control performance of the system, but also cause the system instability. The following is a brief introduction to the various uncertain factors in the network environment.

(1) Time delay

In the network system, each node shares the communication channel in a time-sharing manner. Due to the limited network bandwidth and irregular change of network data flow, when multiple nodes exchange data through the network, time delay of the information exchange is inevitable, which is called as network-induced delay. This is attributed to the waiting time of the signal before transmission. Before the signal is transmitted from one node to another, the node needs to detect the congestion degree of the network. The new signal will be transmitted only when the network is idle. The amount of this waiting time is random, so the signal transmission delay is random. The time delay has a significant effect on the performance of NCS, which not only reduces the performance of the control system, but also causes the system instability.

(2) Packet loss

In the NCS, the causes of packet loss are various, such as the network congestion, connection interruption and transmission failure of node competing data. In addition, during the network transmission process, the packet will be retransmitted when there is an error. If a packet has not been successfully sent within the specified retransmission time, it will be discarded. This can be regarded as a special case of packet loss. The occurrence of packet loss is also random. From the perspective of information transmission, the occurrence of packet loss is regarded as the temporary interruption of information transmission channel. This causes significant changes in the structure and parameters of the system. A stable running NCS allows a certain amount of packet loss. However, when the packet loss rate is beyond a certain value, the system will be unstable.

(3) Single-packet transmission and multi-packet transmission

The transmission of information in the form of packet is a feature of NCS that differ from traditional control systems. The so-called single-packet transmission means that the data to be transmitted is encapsulated and transmitted in one packet. The multi-packet transmission means that the data to be transmitted is divided into a plurality of data packets, and transmitted in time sharing. For MIMO control systems which distributed in spatial (physical time) locations, it is difficult to transmit data from all sensors in a single packet, and then the multi-packet transmission is adopted. Since the network links that transmit packets may be different, multi-packet transmissions may suffer from different delays or even packet losses.

(4) Packet timing disorder

In a network environment, the transmitted data flows through many computers and communication devices, and the transmission path is not unique. This inevitably leads to the packet timing disorder. In the case of a single packet, it means that the arrival timing order of a plurality of complete data arriving at the target node is different from the original timing order. In the case of multiple packets, the arrival timing order of multiple packets arriving at the target node is different from the original timing order. In addition, node collisions, network congestions and connection interruptions result in the packet transmission delay. This kind of network-induced delay, especially when the delay is random and greater than the sampling period of the system, inevitably causes the timing disorder of packet transmission. This packet timing disorder caused by random delay is also random.

(5) Multiple sampling rates

Multiple sampling rates mean that two or more samplers in the control system sample at different sampling periods. For NCS, due to the distribution of nodes and the complexity of control loop, physical signals of different performances are impossible to adopt a uniform sampling rate, and it is impossible to control all actuators with a uniform sampling frequency. Additionally, in a network environment, sensors with low sampling frequency can reduce network traffic more effectively, controllers and actuators with higher sampling frequency can improve the system performance. The smaller the sampling periods of the sampler and the holder are, the better the performance of the system is. However, the A/D and D/A conversion will be faster, which results in the higher cost. For systems with different frequency signals, to obtain better performance and save hardware costs, the natural solution is that the system adopts multiple sampling rates. This means each subsystem adopts different sampling rates based on its functional requirements.

(6) Node driving mode

In NCS, the driving mode of a node refers to the startup mode of the sensor node, the controller node and the actuator node. Currently, the node in NCS has two driving modes: clock-driven and event-driven. Clock-driven means that the network node starts working at a predetermined time, and periodically works. The event-driven means that the network node starts working when a specific event occurs. The driving mode of sensor commonly is clock-driven and controller or actuator is event-driven. This reduces both the waiting time for sampling and the network delay to a certain extent. In addition, it avoids problems during clock-driven, such as the difficult of clock synchronization between nodes, empty sampling and data loss. However, event driven is not easy to implement, and control networks supporting event-driven in applications are few.

(7) Nonlinear interference

There are more or less nonlinear phenomena in the actual system. The nonlinearity is major attributed to two factors. One is the imperfection and inherent characteristics of the system. The other is interference caused by the external environment. For NCS, due to its complex external environment, the interference received by the system is no longer simply white noise or energy bounded signals, usually a more general nonlinear signal. Generally, randomly occurring network attacks are also nonlinear, and the occurrence of such nonlinearities is not fixed. As the external environment varies, nonlinear interference or network attacks acting on the system will vary, and this variation is random. Nonlinear interference or network attacks that occur randomly are one of the issues that should be considered in NCS.

In addition, there are problems such as clock synchronization between nodes, network scheduling and network congestion.

Due to the complexity of the NCS structure and above problems, many assumptions of traditional control theory, such as single rate-sampling, sensing and adjustment without delay, are no longer suitable for NCS. The traditional control theory and method cannot be directly applied to NCS. The interaction between control and communication complicates the analysis and design of NCS that adds the communication network in closed-loop control system. The control system based on the communication network presents new challenges to technologies such as communication, signal processing and control.

1.4 Research status of NCS

NCS is a combination of network technology and control theory. Therefore, the research on NCS has two major directions. One is the control of the network. From the perspective of communication network, researchers pay attention to the improvement of intrinsic characteristics of the network, such as proposing a new type of network communication protocol and network scheduling algorithm, to solve problems of network delay, timing disorder and packet loss. The other one is to control through the network. From the perspective of control theory, based on the existing network structure, protocol, *etc*., the reasonable control structure and control algorithm are designed to reduce adverse effects of time delay and packet loss on the control system. This ensure good control performance of the NCS. Control through the network is the main research direction of this book. Since the advent of NCS, many scholars have studied network control algorithms and obtained fruitful research results. Here, we only give a brief overview about H_∞ method on filtering and control problems.

1.4.1 Research status on H_∞/robust filtering problem

Kalman filtering generally requires that the observed noise obeys the Gaussian distribution, and the statistical properties are known. This cannot be met in practice, thus limiting the application of Kalman filtering and its improved method. H_∞ filtering does not require to know accurately a priori information of noise. In recent years, it has attracted extensive research interest from scholars. A series of valuable results have been obtained for H_∞ filtering of networked systems [10-36].

Literatures [10,11] and [12,13] have studied discrete-time linear systems and nonlinear systems with packet loss problems, respectively. Among them, the literature [10] reported the multiple packet losses of both channels. The literature [11] reported robust weighted H_∞ filtering for a multi-sensor system with uncertain measurements. Literature [12] reported the design of H_∞ filtering with uncertain packet loss for nonlinear systems. Literature [13] reported an output-bounded H_∞ filtering for networked systems with decaying measurement and random nonlinearity. For the problem of random time lag in network transmission, literatures [14,15] studied the networked system with one-step random delay, which was described by random variables of Bernoulli distribution. Literatures [16,17] adopted the Markov chain to represent the multi-step random delay in network transmission, where the transition probabilities are known and the partial transition probabilities are unknown, respectively. Literature [18] discussed the design of H_∞ filtering for the limited communication in networked systems. The above literatures only considered one random phenomenon in network transmission. Commonly, various random phenomena occur simultaneously, so more and more researches focus on simultaneously describing various uncertain phenomena and designing corresponding filters. Literatures [19-22] studied the design of H_∞ filtering for NCS with both the random communication delay and packet loss. Among them, the literature [19] studied the nonlinear system. Literature [20] studied the singular system. Literatures [21,22] considered systems with mixed time delays (time-varying delay and distributed time-delay). Literatures [23-25] investigated the H_∞ filtering for a nonlinear random networked system with sensor saturation. Literature [26] studied the H_∞ filtering design for multi-sensor system with multiple packet losses when sensors had multi-rate sampling. Literature [27] designed a centralized fusion and distributed fusion H_∞ filtering for multi-sensor systems by considering both time delay and missing measurement phenomena. In addition, the fuzzy H_∞ filtering for nonlinear systems described by T-S fuzzy models [28-31] and distributed H_∞ filtering for wireless sensor network systems [32-36] have been reported.

1.4.2 Research status on H_∞/robust control problem

Due to the random nature of the NCS, the uncertainties caused by a series of problems such as environmental changes and component aging during the operation of the control system inevitably exist, the robust /H_∞ control method has been widely used [37-50].

The literature [37] studied the random packet loss phenomenon of the sensor channel and the controller channel, and designed the robust H_∞ controller of the system based on the state observer. Literature [38] built the model of Markov jump process to describe the packet loss of sensor and controller channels, and studied the H_∞ control issue for NCS with unknown partial state-transition probability and analyzed its random stability. In view of the random delay phenomenon, the literature [39] studied the H_∞ state feedback control for serial NCS with time delay of the controller-actuator channel, in which the time delay was uncertain and less than one sampling period. The literatures [40,41] studied the random time delay of sensor channels and controller channels. Among them, the literature [40] built the model about the time delay by using Markov chain and proposed the new H_2 and H_∞ norm to design the robust hybrid H_2/H_∞ controller. Literature [41] reported the influence of data quantization, and proposed a new NCS model with quantization and random time delay. Based on this model, an H_∞ state feedback controller was designed and the relationships among quantization, random time delay and system performance were analyzed. Literatures [42,43] studied the design of corresponding H_∞ controller for the random sampling system. Literature [44] studied the design of H_∞ controller for scalar nonlinear systems. In addition, the H_∞ fuzzy controllers for the nonlinear system described by the T-S fuzzy model [45-47], H_∞ controllers for the sensor network system with random communication topology [48], the robust consensus control [49] and collaborative control [50] of multi-agent systems have been reported.

1.5 Main content and structure about this book

In summary, NCS, as an emerging research field, has achieved remarkable results in filter and controller design after nearly 20 years of development. Because NCS is essentially a random system, random methods are often adopted to model it. There are two main types up to now. One type is modelling the uncertain phenomena in the network using independent and identically distributed Bernoulli random variables. The NCS is transformed into a stochastic parameterized system for the analysis and design. The other one describes the uncertain phenomena by using the Markov chain. The corresponding NCS is transformed into a Markov jump system for the analysis and design. This book mainly uses the modeling method based on the Bernoulli-distributed random variables to describe various random factors in the network, such as the time delay, packet loss, timing disorder, nonlinear interference, *etc*., in a unified framework. Consequently, the stochastic parameterized method is used for the analysis and design for the filter and the controller.

The book is divided into three parts. The first part is the theoretical part of the NCS. It introduces the development history, uncertain factors and main research status of NCS. The second part is about the design of H_∞ filtering. It contains Chapter 2 to Chapter 5. In Chapter 2, H_∞ filtering is investigated for linear systems by considering multiple packet dropouts and random delays in network transmission. In Chapter 3, H_∞ filtering design problem is considered for multiple-channel NCS with not only consecutive packet losses and varying delays but also randomly occurred nonlinearities. In Chapter 4, H_∞ filtering is studied for NCS with delayed measurements, packet losses and randomly varying nonlinearities in a multiple packets compensation strategy. In Chapter 5, H_∞ filtering is dealt with for fuzzy-model-based nonlinear NCS subject to sensor saturations. The third part discusses the design of H_∞ controller. It contains Chapter 6 to Chapter 8. In Chapter 6, H_∞ control problem is considered for NCS with one-step random delays and packet dropouts. In Chapter 7, H_∞ control is designed based on observer for NCS with consecutive packet dropouts and multiple packet dropouts. In Chapter 8, H_∞ control is investigated where the NCS is stochastic nonlinear and the randomly occurred sensor saturations, multiple delays and packet dropouts are considered. At the end of each chapter, the MATLAB simulation results are given for demonstrating the effectiveness of the proposed method.

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Chapter 2 H_∞ Filtering for Networked Linear Systems with Multiple Packet Dropouts and Random Delays

2.1 Introduction

With the rapid development of computer and communication technology, networks have been widely used as the medium in modern engineering systems to connect the spatially distributed sensors, actuators and controllers or filters. Such systems are the so-called networked control systems (NCSs) which have many advantages, such as low cost, simple installation and maintenance, convenient system diagnosis and increased system agility [1]. However, due to the limited bandwidths of the communication channels, the network-induced delays are inevitable during the data transmission through the networks from the sender to the receiver [2,3]. A transmission delay may be less than or larger than one sampling period which is the so-called short time-delay or large time-delay. For example, the TCP/IP communication protocol has the large communication delays but the UDP/IP has generally the short communication delays [4]. The network-induced delays have the random characteristic in nature but are bounded. If the packet is with a delay longer than a certain pre-determined number, one possible strategy is to discard the packet and treat it as a packet dropout, which could deteriorate the system performance or even leads to the instability. So, the random delays and packet dropouts are the two important issues which have attracted considerable research attention in the NCSs.

The filtering problem for networked systems has been a focus of research due to their important engineering applications such as target tracking, signal processing and control application [5-8]. It is well known that the Kalman filtering is the classical scheme to deal with the estimation problem effectively. In a network environment, the Kalman filtering should be modified to conduct the random phenomena [9-12]. However, one primary limitation of Kalman filtering is that the external disturbances are required to be Gaussian noises with known statistical property. Such a requirement is not always satisfied in practical applications. For this case, the H_∞ method is taken as the alternative method. A great number of important results for the H_∞ control problems have been reported [13-16]. As for H_∞ filtering problem, the objective of it is to minimize the highest energy gain of the estimation error for all initial conditions and noises where the noise signals are assumed to be arbitrary but with bounded energy or bounded average power rather than just Gaussian. Hence, the H_∞ filtering problem of networked systems has also received extensive research attention [17-22].

Due to the random nature of transmission delays and packet dropouts, they can be described by Bernoulli distributed white sequences or Markov chains [23-25]. As for the first type, a set of Bernoulli distributed random variables are used to establish the NCSs with multiple packet dropouts and random transmission delays simultaneously [26,27]. However, in [26], the model deals with the networks with retransmission mechanism which may cause the network congestion. A new model is established to describe the phenomena of multiple random delays and packet dropouts in [27] and the optimal linear filters with/without time stamps have been designed by the innovation analysis approach, which is generalized to multi-sensor systems with different delay and loss rates in [28] where a distributed fusion filter is designed. However, the exogenous disturbance of the system in [27,28] is required to be stochastic Gaussian noise with the known statistical information. When a priori information on the external noise is not available, the designed scheme is no longer applicable. In [29], the full-order H_∞ filter is designed based on the model of [27] to overcome the requirement of Gaussian noise. However, due to the effect of the multi-step time delay, the inequality scaling is used during the LMI transformation. So, the obtained result is conservative. In [30], the H_∞ filtering problem is investigated by the delay partitioning method for the NCSs with time varying delay where d_k denotes the varying delays and is with lower and upper bounds $d_m \leq d_k \leq d_M$. However, the lower bound of delay d_m can not be zero and the Lyapunov-Krasovskii functional candidate is too complex in this method. In recent literature [31], a new method is presented to deal with the bounded delay where the lower bound d_m can be equal to zero. But the data packet dropouts are not considered simultaneously.

In this chapter, the H_∞ filtering problem is taken into account for discrete-time linear systems with random packet dropouts and time delays simultaneously. By introducing some new variables, the original system can be modeled as a stochastic parameterized system and then the filtering error system can be constructed. Sufficient conditions are proposed based on the Lyapunov stability analysis and LMI technique such that the derived filtering error system is mean-square exponentially stable with a prescribed disturbance attenuation level. An F-404 aircraft engine system is utilized to illustrate the effectiveness and applicability of the proposed algorithms. The main contributions are as follows: (1) By using a set of Bernoulli distributed random variables, a unified framework is established to describe the random phenomena of the NCS including consecutive packet dropouts and multiple transmission delays. (2) Based on the definition of new variables, a state augmentation method is presented to deal with the effect of multi-step time-delay. Sufficient conditions are derived through Lyapunov stability analysis for the filtering error system to be mean-square exponentially stable and to achieve a prescribed H_∞ performance level. (3) A simple LMI approach is exploited and solvability of the desired filter. Moreover, the conservatism can be reduced compared with the full-order filter design approach by inequality scaling during the LMI transformation. This makes the structure of the filter problem as more general one for the NCS with multiple packet dropouts and random delays.

2.2 Problem formulation

Consider the discrete-time network-based linear system:

$$\begin{cases} x_{k+1} = Ax_k + Bw_k \\ \tilde{y}_k = Cx_k + Dw_k \\ z_k = Lx_k \end{cases} \tag{2.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $\tilde{y}_k \in \mathbb{R}^r$ is the measured output, $z_k \in \mathbb{R}^m$ is the signal to be estimated, $w_k \in \mathbb{R}^q$ is the exogenous disturbance input which belongs to $l_2[0,\infty)$, and A, B, C, D and L are known constant matrices with appropriate dimensions.

In the following discussion, we assume that the sensor and filter are clock-driven, and the sensor sampling, sending and the filter receiving are synchronous. When the data packets are transmitted through the networks from the sensor to the remote filter, the phenomena of transmission delays and packet dropouts unavoidably occur. Thus, the remote filter will receive the incomplete sensor information. The model to describe the network-induced issues is given by [27]

$$\begin{aligned} y_k = {} & \alpha_{0,k}\tilde{y}_k + (1-\alpha_{0,k})\{(1-\alpha_{0,k-1})\alpha_{1,k}\tilde{y}_{k-1} \\ & +[1-(1-\alpha_{0,k-1})\alpha_{1,k}]\{(1-\alpha_{0,k-2})(1-\alpha_{1,k-1})\alpha_{2,k}\tilde{y}_{k-2} + \cdots \\ & +[1-\textstyle\prod_{i=0}^{d-2}(1-\alpha_{i,k-d+i+1})\,\alpha_{d-1,k}]\textstyle\prod_{i=0}^{d-1}(1-\alpha_{i,k-d+i})\,\alpha_{d,k}\tilde{y}_{k-d}\}\}\cdots\} \end{aligned} \tag{2.2}$$

where $y_k \in \mathbb{R}^r$ is the measurement received by the remote filter, $\alpha_{i,k}(i = 0,1,\cdots,d)$ are Bernoulli distributed random variables which are uncorrelated with each other and satisfy the probabilities $\text{Prob}\{\alpha_{i,k} = 1\} = \bar{\alpha}_i$ and $\text{Prob}\{\alpha_{i,k} = 0\} = 1 - \bar{\alpha}_i$, where

$0 \leq \bar{\alpha}_i \leq 1$, and d is the largest transmission delay from the sensor to the filter.

The structure of the considered networked system is illustrated in Figure 2.1.

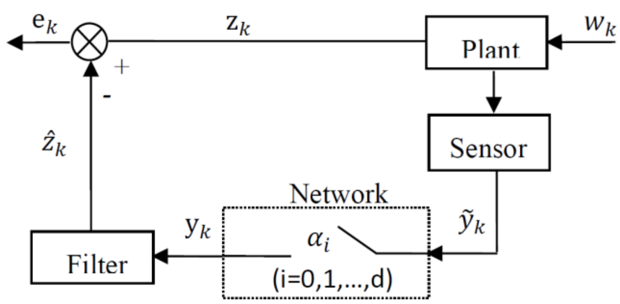

Figure 2.1 Diagram of the networked system.

Remark 2.1 To avoid network congestion as possible, a packet at the sensor side is only sent once. The filter only receives one packet or nothing at each time, which can be met in the case of single-radio at the filter side.

Model (2.2) describes the possible one-step, two-step up to d-step transmission delays and packet dropouts where the delays are randomly varying and packet dropouts are possibly consecutive. Take $d = 2$ as an example, the measurement received by the filter can be given as follows:

$$y_k = \alpha_{0,k}\tilde{y}_k + (1-\alpha_{0,k})\{(1-\alpha_{0,k-1})\alpha_{1,k}\tilde{y}_{k-1}$$
$$+[1-(1-\alpha_{0,k-1})\alpha_{1,k}](1-\alpha_{0,k-2})(1-\alpha_{1,k-1})\alpha_{2,k}\tilde{y}_{k-2}\}$$

The data transmission case for $d = 2$ is given in the following Table 2.1.

From Table 2.1, we can see that $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_5$, $\tilde{y}_9$ and $\tilde{y}_{10}$ are received on time, $\tilde{y}_6$ is delayed one step, $\tilde{y}_4$ is delayed two step, $\tilde{y}_7$ and $\tilde{y}_8$ are lost. Moreover, it can be readily obtained that the on-time received rate is $\bar{\alpha}_0$, the one-step delay rate is $(1-\bar{\alpha}_0)^2\bar{\alpha}_1$, the two-step delay rate is $(1-\bar{\alpha}_0)^2[1-(1-\bar{\alpha}_0)\bar{\alpha}_1]((1-\bar{\alpha}_1)\bar{\alpha}_2$, and the packet dropout rate is $1-\bar{\alpha}_0-(1-\bar{\alpha}_0)^2\bar{\alpha}_1-(1-\bar{\alpha}_0)^2[1-(1-\bar{\alpha}_0)\bar{\alpha}_1]((1-\bar{\alpha}_1)\bar{\alpha}_2$.

Table 2.1 Data transmission in networks.

k	1	2	3	4	5	6	7	8	9	10
$\alpha_{0,k}$	1	1	0	0	1	0	0	0	1	1
$\alpha_{1,k}$	-	-	-	1	0	0	1	0	0	-
$\alpha_{2,k}$	-	-	-	-	-	1	-	-	0	-
y_k	$\tilde{y}_1$	$\tilde{y}_2$	0	$\tilde{y}_3$	$\tilde{y}_5$	$\tilde{y}_4$	$\tilde{y}_6$	0	$\tilde{y}_9$	$\tilde{y}_{10}$

Remark 2.2 From the above analysis, it is obvious that the time delay rates and packet dropout rates can be calculated by $\bar{\alpha}_i$. That is to say, the probabilities $\bar{\alpha}_i$ are known is equivalent to that the delay and packet dropout rates are known. In a practical network, we can obtain the on-time rate, time delay rate and packet dropout rate by using time-stamp technique and statistics method, from which the $\bar{\alpha}_i$ can be computed. So, the probability-depended method is applied in the subsequent part to obtain the main results.

In the following, some new variables are introduced for the derivation.

Let $\xi_{j,k} = \prod_{i=0}^{j-1}(1-\alpha_{i,k+i})\alpha_{j,k+j}$, $j = 0,1,\cdots,d$ with $\xi_{0,k} = \alpha_{0,k}$ and $\tilde{Y}_{j,k} = \xi_{j,k}\tilde{y}_k + (1-\xi_{j,k})\tilde{Y}_{j+1,k-1}, j = 1,2,\cdots,d-1$; $\tilde{Y}_{d,k} = \xi_{d,k}\tilde{y}_k$.

Then, by defining an augmented state $X_{k+1} = \begin{bmatrix} x_{k+1}^T & \tilde{Y}_{1,k}^T & \tilde{Y}_{2,k}^T & \cdots & \tilde{Y}_{d,k}^T \end{bmatrix}^T$ we can obtain a compact form of system (2.1)-(2.2) as follows:

$$\begin{cases} X_{k+1} = \tilde{A}_k X_k + \tilde{B}_k w_k \\ \tilde{y}_k = \tilde{C}_k X_k + \tilde{D}_k w_k \\ z_k = L_c X_k \end{cases} \tag{2.3}$$

where

$$\tilde{A}_k = A_c + \textstyle\sum_{j=1}^{d}\xi_{j,k}A_j,\ \tilde{B}_k = B_c + \sum_{j=1}^{d}\xi_{j,k}B_j,\ \tilde{C}_k = C_c + \xi_{0,k}C_0,\ \tilde{D}_k = \xi_{0,k}D \tag{2.4}$$

with A_c, B_c, A_j, B_j , $j = 1,2,\cdots,d$, C_c, C_0 and L_c are defined as follows:

$$A_c = \begin{bmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad A_j = \underset{\substack{\downarrow \\ (j+2)\,block\ column}}{\begin{bmatrix} 0 & \cdots & 0 \\ C & 0 & -I \\ 0 & \cdots & 0 \end{bmatrix}} \rightarrow (j+1)\ block\ row \qquad j = 1,2,\cdots,d-1$$

$$A_d = \begin{bmatrix} 0 & 0 \\ C & 0 \\ 0 & 0 \end{bmatrix} \rightarrow (j+1)\ block\ row\ ,\ B_c = \begin{bmatrix} B \\ 0 \end{bmatrix},\ B_j = \begin{bmatrix} 0 \\ D \\ 0 \end{bmatrix} \rightarrow (j+1)\ block\ row,\ j = 1,2,\cdots,d$$

$$C_c = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \end{bmatrix},\ C_0 = \begin{bmatrix} C & -I & 0 & \cdots & 0 \end{bmatrix},\ L_c = \begin{bmatrix} L & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Remark 2.3 Up to now, we have constructed the system (2.3) from (2.1)-(2.2) by introducing a group of new variables $\xi_{j,k}(j = 0,1,\cdots,d)$. Notice that the system (2.3) is a state augmentation form due to the variables $\tilde{Y}_{j,k}$ $(j = 1,2,\cdots,d)$. Although the dimension becomes larger than original system, it can overcome the relativity of the $\alpha_{j,k}(j = 0,1,\cdots,d)$ when the full-order filter design approach. Hence, it can reduce the conservatism in the subsequent LMI-based derivation of the filter design.

Remark 2.4 Note that the system (2.3) is parameterized by stochastic parameters $\xi_{j,k}(j = 0,1,\cdots,d)$ the statistical properties of which are given as follows:

$$E\{\xi_{0,k}\} = \bar{\xi}_0 = \bar{\alpha}_0; E\{\xi_{j,k}\} = \bar{\xi}_j = \prod_{i=0}^{j-1}(1-\bar{\alpha}_i)\,\bar{\alpha}_j,\ E\{(\xi_{j,k}-\bar{\xi}_j)^2 = \bar{\xi}_j(1-\bar{\xi}_j)\},\ j = 1,2,\cdots,d;$$

$$E\{\xi_{j,k}\xi_{l,k}\} = 0,\ j \neq l;\ E\{\xi_{j,k}\xi_{l,t}\} = \bar{\xi}_j\bar{\xi}_l,\ \mathrm{k} \neq t,\ j,l = 0,1,\cdots,d;$$

$$E\{(\xi_{j,k}-\bar{\xi}_j)(\xi_{l,k}-\bar{\xi}_l) = -\bar{\xi}_j\bar{\xi}_l; j \neq l.$$

Now, we are interested in obtaining the estimate of the signal z_k from the measurement $y_k \in \mathbb{R}^r$ received by the filter. The filter to be designed is of the following form:

$$\begin{cases} \hat{X}_{k+1} = A_f\hat{X}_k + B_f y_k \\ \hat{z}_k = C_f\hat{X}_k + D_f y_k \end{cases} \tag{2.5}$$

where $\hat{X}_k \in \mathbb{R}^{n+rd}$ is the estimate of the augmented state X_k and $\hat{z}_k \in \mathbb{R}^m$ is the estimate of the signal z_k. A_f, B_f, C_f and D_f are the coefficient matrices of the filter to be designed.

Define the filtering error $e_k = z_k - \hat{z}_k$. Combining (2.3) and (2.5), we obtain the filtering error system as follows:

$$\begin{cases} \eta_{k+1} = \widetilde{\Phi}_k\eta_k + \widetilde{\Gamma}_k w_k \\ e_k = \widetilde{\mathrm{H}}_k\eta_k + \widetilde{\mathrm{G}}_k w_k \end{cases} \tag{2.6}$$

where $\eta_k = [X_k^T \quad \hat{X}_k^T]^T$, $\eta_k = [X_k^T \quad \hat{X}_k^T]^T$, $\widetilde{\Phi}_k = \begin{bmatrix} \widetilde{\mathrm{A}}_k & 0 \\ B_f\widetilde{\mathrm{C}}_k & A_f \end{bmatrix}$, $\widetilde{\Gamma}_k = \begin{bmatrix} \widetilde{\mathrm{B}}_k \\ B_f\widetilde{\mathrm{D}}_k \end{bmatrix}$, $\widetilde{\mathrm{H}}_k = \begin{bmatrix} L_c - D_f\widetilde{\mathrm{C}}_k & -C_f \end{bmatrix}$ and $\widetilde{\mathrm{G}}_k = -D_f\widetilde{\mathrm{D}}_k$.

From (2.3) and (2.4), we can rewrite

$$\widetilde{\Phi}_k = \Phi_c + \textstyle\sum_{j=0}^{d}\xi_{j,k}\,\Phi_j,\ \widetilde{\Gamma}_k = \Gamma_c + \sum_{j=0}^{d}\xi_{j,k}\,\Gamma_j,\ \widetilde{H}_k = H_c + \xi_{0,k}H_0,\ \widetilde{G}_k = \xi_{0,k}G_0 \tag{2.7}$$

where $\Phi_c, \Gamma_c, H_c,\ H_0,\ G_0, \Phi_j$ and $\Gamma_j\ j = 0,1,\cdots,d$ are defined by

$$\Phi_c = \begin{bmatrix} A_c & 0 \\ B_fC_c & A_f \end{bmatrix},\ \Phi_0 = \begin{bmatrix} 0 & 0 \\ B_fC_0 & 0 \end{bmatrix},\ \Phi_j = \begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix},\ j = 1,2,\cdots,d,\ \Gamma_c = \begin{bmatrix} B_c \\ 0 \end{bmatrix},$$

$$\Gamma_0 = \begin{bmatrix} 0 \\ B_fD \end{bmatrix},\ \Gamma_j = \begin{bmatrix} B_j \\ 0 \end{bmatrix},\ j = 1,2,\cdots,d,\ H_c = [L_c - D_fC_c \quad -C_f],$$

$$H_0 = [-D_fC_0 \quad 0],\ G_0 = -D_fD$$

For the convenience of our later discussion, we give the following notations: $\bar{A} = E\{\widetilde{\mathrm{A}}_k\}$, $\bar{B} = E\{\widetilde{\mathrm{B}}_k\}$, $\bar{C} = E\{\widetilde{\mathrm{C}}_k\}$, $\bar{D} = E\{\widetilde{\mathrm{D}}_k\}$, $\bar{\Phi} = E\{\widetilde{\Phi}_k\}$, $\bar{\Gamma} = E\{\widetilde{\Gamma}_k\}$, $\bar{H} = E\{\widetilde{\mathrm{H}}_k\}$, and $\bar{G} = E\{\widetilde{\mathrm{G}}_k\}$ which are given in (2.4) and (2.7), respectively, where $\xi_{j,k}(j = 0,1,\cdots,d)$ are replaced by their expectations $\bar{\xi}_j, (j = 0,1,\cdots,d)$.

Definition 2.1 For a given scalar $\gamma > 0$, the filtering error system (2.6) is said to be exponentially stable in the mean square sense with an H_∞ performance level γ if the following conditions hold:

(i) The filtering error system (2.6) is said to be exponentially stable in the mean square under $w_k = 0$, if there exist constants $\varphi > 0$ and $\tau \in (0,1)$, such that

$$E\{\|\eta_k\|^2\} \leq \varphi\tau^k E\{\|\eta_0\|^2\}, \text{ for all } \eta_0 \in \mathbb{R}^{2(n+rd)}, k \in I^+.$$

(ii) The filtering error system (2.6) is said to be satisfied with the H_∞ performance constraint if under zero initial condition and for any non-zero $w_k \in l_2[0,\infty)$, the filtering error e_k satisfies

$$\sum_{k=0}^{\infty} E\{\|e_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\|w_k\|^2\}. \tag{2.8}$$

In the subsequent, we aim to design a linear filter of the form (2.5) for system (2.1) such that the filtering error system (2.6) is exponentially stable in the mean square with an H_∞ performance level $\gamma > 0$ in the presence of the multiple packet dropouts, random delays, and exogenous disturbance.

2.3 Filtering performance analysis

In this section, we shall establish a sufficient condition of mean-square exponential stability with H_∞ performance for the filtering error system (2.6), which will be fundamental in the derivation of our H_∞ filter.

Theorem 2.1 Let the filter parameters A_f, B_f, C_f and D_f be given and γ be a pre-specified positive constant. Then the filtering error system is exponentially stable in the mean square sense with a guaranteed H_∞ filtering performance γ, if there exists matrix $P > 0$ such that

$$\begin{bmatrix}
-P & * & * & * & * & * & * & * & * & * & * \\
0 & -\gamma^2 I & * & * & * & * & * & * & * & * & * \\
\bar{\Phi} & \bar{\Gamma} & -P^{-1} & * & * & * & * & * & * & * & * \\
\theta_0\Phi_0 & \theta_0\Gamma_0 & 0 & -P^{-1} & * & * & * & * & * & * & * \\
\vdots & \vdots & 0 & 0 & \ddots & * & * & * & * & * & * \\
\theta_d\Phi_d & \theta_d\Gamma_d & 0 & 0 & 0 & -P^{-1} & * & * & * & * & * \\
\theta_{0,1}\Phi_{0,1} & \theta_{0,1}\Gamma_{0,1} & 0 & 0 & 0 & 0 & -P^{-1} & * & * & * & * \\
\vdots & \vdots & 0 & 0 & 0 & 0 & 0 & \ddots & * & * & * \\
\theta_{d-1,d}\Phi_{d-1,d} & \theta_{d-1,d}\Gamma_{d-1,d} & 0 & 0 & 0 & 0 & 0 & 0 & -P^{-1} & * & * \\
\bar{H} & \bar{G} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & * \\
\beta_0 H_0 & \beta_0 G_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \tag{2.9}$$

where

$$\Phi_{i,j} = \Phi_i - \Phi_j,\ \Gamma_{i,j} = \Gamma_i - \Gamma_j,\ 0 \leq i < j \leq d$$

$$\beta_0 = \sqrt{\bar{\xi}_0(1-\bar{\xi}_0)},\ \theta_i = \sqrt{\bar{\xi}_i\left(1-\sum_{j=0}^{d}\bar{\xi}_j\right)},\ i = 0,1,\cdots,d; \tag{2.10}$$

$$\theta_{i,j} = \sqrt{\bar{\xi}_i\bar{\xi}_j},\ \leq i < j \leq d.$$

Proof The proof is divided into the following two parts: (i) prove that the filtering error system (2.6) is mean square exponentially stable in the absence of w_k; and (ii) prove that the condition (2.8) holds under zero initial condition for any non-zero w_k.

(i) We first prove the mean-square exponential stability of the filtering error system (2.6) with $w_k = 0$.

Select the Lyapunov functional candidate as

$$V_k(\eta_k) = \eta_k^T P \eta_k \tag{2.11}$$

Then

$$\Delta V_k(\eta_k) = E\{V_{k+1}(\eta_{k+1})|\eta_k\} - V_k(\eta_k) = E\{\eta_k^T(\widetilde{\Phi}_k^T P \widetilde{\Phi}_k - P)\eta_k\}$$

where $\widetilde{\Phi}_k$ can be rewritten as the following form:

$$\widetilde{\Phi}_k = \bar{\Phi} + \sum_{i=0}^{d}(\xi_{i,k} - \bar{\xi}_i)\Phi_i \tag{2.12}$$

We have

$$\Delta V_k(\eta_k) = E\{\eta_k^T(\widetilde{\Phi}_k^T P \widetilde{\Phi}_k - P)\eta_k\} = \eta_k^T[(\bar{\Phi}^T P \bar{\Phi} - P) + \textstyle\sum_{i=0}^{d}(\xi_{i,k} - \bar{\xi}_i)^2\, \Phi_i^T P \Phi_i]$$
$$+ \textstyle\sum_{i=0}^{d-1}\sum_{j=i+1}^{d} E\{(\xi_{i,k} - \bar{\xi}_i)(\xi_{j,k} - \bar{\xi}_j)\}\,(\Phi_i^T P \Phi_j + \Phi_j^T P \Phi_i)\eta_k]$$

By using the relation of

$$-\Phi_i^T P \Phi_j - \Phi_j^T P \Phi_i = (\Phi_i - \Phi_j)^T P (\Phi_i - \Phi_j) - \Phi_i^T P \Phi_i - \Phi_j^T P \Phi_j$$

and from Remark 2.3, we can obtain

$$\begin{aligned}\Delta V_k(\eta_k) = E\{\eta_k^T(\widetilde{\Phi}_k^T P \widetilde{\Phi}_k - P)\eta_k\} = \ &\eta_k^T[(\bar{\Phi}^T P \bar{\Phi} - P) + \textstyle\sum_{i=0}^{d}\theta_i^2\, \Phi_i^T P \Phi_i + \\ &\textstyle\sum_{i=0}^{d-1}\sum_{j=i+1}^{d}\theta_{i,j}^2\, \Phi_{i,j}^T P \Phi_{i,j}]\eta_k = \eta_k^T \Omega \eta_k\end{aligned} \tag{2.13}$$

where $\Omega = (\bar{\Phi}^T P \bar{\Phi} - P) + \sum_{i=0}^{d}\theta_i^2\, \Phi_i^T P \Phi_i + \sum_{i=0}^{d-1}\sum_{j=i+1}^{d}\theta_{i,j}^2\, \Phi_{i,j}^T P \Phi_{i,j}$, $\theta_{i,j}, \theta_i$ and $\Phi_{i,j}$ are defined by (2.10).

By Shur complement, $\Omega < 0$ is equivalent to

$$\begin{bmatrix} -P & * & * & * & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ \bar{\Phi} & \bar{\Gamma} & -P^{-1} & * & * & * & * & * & * \\ \theta_0\Phi_0 & \theta_0\Gamma_0 & 0 & -P^{-1} & * & * & * & * & * \\ \vdots & \vdots & 0 & 0 & \ddots & * & * & * & * \\ \theta_d\Phi_d & \theta_d\Gamma_d & 0 & 0 & 0 & -P^{-1} & * & * & * \\ \theta_{0,1}\Phi_{0,1} & \theta_{0,1}\Gamma_{0,1} & 0 & 0 & 0 & 0 & -P^{-1} & * & * \\ \vdots & \vdots & 0 & 0 & 0 & 0 & 0 & \ddots & * \\ \theta_{d-1,d}\Phi_{d-1,d} & \theta_{d-1,d}\Gamma_{d-1,d} & 0 & 0 & 0 & 0 & 0 & 0 & -P^{-1} \end{bmatrix} < 0 \tag{2.14}$$

Subsequently, $\Delta V_k(\eta_k) < -\lambda_{min}(-\Omega)\eta_k^T\eta_k < -\phi\eta_k^T\eta_k$, where $0 < \phi < \min\{\lambda_{min}(-\Omega), \lambda_{max}(P)\}$, thus, we can confirm from Lemma 1 in [14] that the filtering error system (2.6) is mean-square exponentially stable. Obviously, (2.9) implies (2.14). So, the first part of the proof is completed.

(ii) Next, we pay our attention to proving that for all $w_k \neq 0$, the filtering error system (2.6) satisfies (2.8) under zero initial condition. In this case, we introduce the following index:

$$J = E\{\sum_{k=0}^{\infty}[e_k^T e_k - \gamma^2 w_k^T w_k]\} \tag{2.15}$$

Under zero initial condition, $V_0(\eta_0) = 0$, we can obtain

$$J \le E\{\sum_{k=0}^{\infty} e_k^T e_k - \gamma^2 w_k^T w_k + \Delta V_k(\eta_k)\}.$$

if

$$E\{\Delta V_k(\eta_k) + e_k^T e_k - \gamma^2 w_k^T w_k\} < 0 \tag{2.16}$$

Summing up (2.16) from 0 to ∞ with respect to k, then we have $J < 0$, that is $\sum_{k=0}^{\infty} E\{\|e_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\|w_k\|^2\}$ which means (2.8). Now, we focus on the proof of (2.16). Let us consider

$$\begin{aligned} &E\{\Delta V_k(\eta_k) + e_k^T e_k - \gamma^2 w_k^T w_k\} = E\{\eta_{k+1}^T P \eta_{k+1} - \eta_k^T P \eta_k + e_k^T e_k - \gamma^2 w_k^T w_k\} \\ &= E\{[\tilde{\Phi}_k \eta_k + \tilde{\Gamma}_k w_k]^T P [\tilde{\Phi}_k \eta_k + \tilde{\Gamma}_k w_k]\} - \eta_k^T P \eta_k + E\{[\tilde{\mathrm{H}}_k \eta_k + \tilde{\mathrm{G}}_k w_k]^T P [\tilde{\mathrm{H}}_k \eta_k \\ &\quad + \tilde{\mathrm{G}}_k w_k]\} - \gamma^2 w_k^T w_k \end{aligned} \tag{2.17}$$

By a similar derivation method to part 1, and using the relation of

$$\begin{aligned} E\left\{\eta_k^T (\tilde{\Phi}_k - \bar{\Phi})^T P (\tilde{\Phi}_k - \bar{\Phi}) \eta_k\right\} &= \eta_k^T [\textstyle\sum_{i=0}^{d} \theta_i^2 \Phi_i^T P \Phi_i + \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} \theta_{i,j}^2 \Phi_{i,j}^T P \Phi_{i,j}] \eta_k, \\ E\left\{w_k^T (\tilde{\Gamma}_k - \bar{\Gamma})^T P (\tilde{\Gamma}_k - \bar{\Gamma}) w_k\right\} &= w_k^T [\textstyle\sum_{i=0}^{d} \theta_i^2 \Gamma_i^T P \Gamma_i + \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} \theta_{i,j}^2 \Gamma_{i,j}^T P \Gamma_{i,j}] w_k, \\ E\left\{\eta_k^T (\tilde{\Phi}_k - \bar{\Phi})^T P (\tilde{\Gamma}_k - \bar{\Gamma}) w_k\right\} &= \eta_k^T [\textstyle\sum_{i=0}^{d} \theta_i^2 \Phi_i^T P \Gamma_i + \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} \theta_{i,j}^2 \Phi_{i,j}^T P \Gamma_{i,j}] w_k, \\ E\left\{w_k^T (\tilde{\Gamma}_k - \bar{\Gamma})^T P (\tilde{\Phi}_k - \bar{\Phi}) \eta_k\right\} &= w_k^T [\textstyle\sum_{i=0}^{d} \theta_i^2 \Gamma_i^T P \Phi_i + \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} \theta_{i,j}^2 \Gamma_{i,j}^T P \Phi_{i,j}] \eta_k, \end{aligned} \tag{2.18}$$

We can obtain that

$$E\{\Delta V_k(\eta_k) + e_k^T e_k - \gamma^2 w_k^T w_k\} = \varsigma_k^T \bar{\Omega} \varsigma_k \tag{2.19}$$

where $\varsigma_k = [\eta_k^T \quad w_k^T]^T$, and $\bar{\Omega}$ can be written as the structure of $\bar{\Omega} = \Theta - \begin{bmatrix} \Upsilon_1^T \\ \Upsilon_2^T \end{bmatrix} \tilde{P}^{-1} [\Upsilon_1 \quad \Upsilon_2]$ with $\tilde{P} = diag(\underbrace{-P, -P, \cdots, -P, -P, \cdots, -P}_{d+2+\frac{d(d+1)}{2}}, -I, -I)$, $\Theta =$ diag$(-\mathrm{P}, -\gamma^2)$,

$$\Upsilon_1 = \begin{bmatrix} \bar{\Phi}^T & \theta_0 \Phi_0^T & \cdots & \theta_d \Phi_d^T & \theta_{0,1} \Phi_{0,1}^T & \cdots & \theta_{d-1,d} \Phi_{d-1,d}^T & \bar{\mathrm{H}}^T & \beta_0 \mathrm{H}_0^T \end{bmatrix}^T,$$

$$\Upsilon_2 = \begin{bmatrix} \bar{\Gamma}^T & \theta_0 \Gamma_0^T & \cdots & \theta_d \Gamma_d^T & \theta_{0,1} \Gamma_{0,1}^T & \cdots & \theta_{d-1,d} \Gamma_{d-1,d}^T & \bar{\mathrm{G}}^T & \beta_0 \mathrm{G}_0^T \end{bmatrix}^T$$

Then by Shur complement, (2.9) implies $\bar{\Omega} < 0$. That is, (2.16) holds. The proof is completed.

2.4 H_∞ filter design

In this section, we focus on the filter design problem based on Theorem 2.1. A sufficient condition for the existence of the proposed filter (2.5) is provided in the following theorem.

Theorem 2.2 Given a scalar $\gamma > 0$. There exists a filter in the form of (2.5) such that the filtering error system (2.6) is exponentially stable in the mean square with the H_∞ filtering performance γ, if there exist matrices $X > 0$, $Z > 0$ and matrices $\breve{A}_f$, $\breve{B}_f$, $\breve{C}_f$, $\breve{D}_f$ such that the following LMI holds:

$$\begin{bmatrix} -\Xi & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \bar{\Psi}_{11} & \bar{\Psi}_{12} & \bar{\Xi}_1 & * \\ \bar{\Psi}_{21} & \bar{\Psi}_{22} & 0 & \bar{\Xi}_2 \end{bmatrix} < 0 \tag{2.20}$$

where

$$\begin{aligned} \bar{\Psi}_{11} &= \begin{bmatrix} \Psi_{\bar{\Phi}}^T & \Psi_{\Phi_0}^T & \Psi_{\Phi_1}^T & \cdots & \Psi_{\Phi_d}^T \end{bmatrix}^T, \\ \bar{\Psi}_{12} &= \begin{bmatrix} \Psi_{\bar{\Gamma}}^T & \Psi_{\Gamma_0}^T & \Psi_{\Gamma_1}^T & \cdots & \Psi_{\Gamma_d}^T \end{bmatrix}^T, \\ \bar{\Psi}_{21} &= \begin{bmatrix} \Psi_{\Phi_{0,1}}^T & \cdots & \Psi_{\Phi_{0,d}}^T & \Psi_{\Phi_{1,2}}^T \cdots & \Psi_{\Phi_{d-1,d}}^T & \Psi_{\bar{H}}^T & \Psi_{H_0}^T \end{bmatrix}^T, \\ \bar{\Psi}_{22} &= \begin{bmatrix} \Psi_{\Gamma_{0,1}}^T & \cdots & \Psi_{\Gamma_{0,d}}^T & \Psi_{\Gamma_{1,2}}^T \cdots & \Psi_{\Gamma_{d-1,d}}^T & \Psi_{\bar{G}}^T & \Psi_{G_0}^T \end{bmatrix}^T, \\ \bar{\Xi}_1 &= diag(\underbrace{-\Xi, -\Xi, \cdots, -\Xi}_{d+2}), \\ \bar{\Xi}_2 &= diag(\underbrace{-\Xi, -\Xi, \cdots, -\Xi}_{\frac{d(d+1)}{2}}, -I, -I). \end{aligned} \tag{2.21}$$

with

$$\Xi = \begin{bmatrix} X & Z \\ Z & Z \end{bmatrix}, \ \Psi_{\bar{\Phi}} = \begin{bmatrix} X\bar{A} + \breve{B}_f\bar{C} & X\bar{A} + \breve{B}_f\bar{C} + \breve{A}_f \\ Z\bar{A} & Z\bar{A} \end{bmatrix},$$

$$\Psi_{\Phi_0} = \theta_0 \begin{bmatrix} \breve{B}_f C_0 & \breve{B}_f C_0 \\ 0 & 0 \end{bmatrix}, \ \Psi_{\Phi_i} = \theta_i \begin{bmatrix} XA_i & XA_i \\ ZA_i & ZA_i \end{bmatrix}, \ i = 1,2,\cdots,d,$$

$$\Psi_{\Phi_{0,i}} = \theta_{0,i} \begin{bmatrix} \breve{B}_f C_0 - XA_i & \breve{B}_f C_0 - XA_i \\ -ZA_i & -ZA_i \end{bmatrix}, i = 1,2,\cdots,d,$$

$$\Psi_{\Phi_{i,j}} = \theta_{i,j} \begin{bmatrix} X(A_i - A_j) & X(A_i - A_j) \\ Z(A_i - A_j) & Z(A_i - A_j) \end{bmatrix}, \ 1 \le i < j \le d,$$

$$\Psi_{\bar{\Gamma}} = \begin{bmatrix} X\bar{B} + \breve{B}_f\bar{D} \\ Z\bar{B} \end{bmatrix}, \ \Psi_{\Gamma_0} = \theta_0 \begin{bmatrix} \breve{B}_f\bar{D} \\ 0 \end{bmatrix}, \ \Psi_{\Gamma_i} = \theta_i \begin{bmatrix} XB_i \\ ZB_i \end{bmatrix}, i = 1,2,\cdots,d,$$

$$\Psi_{\Gamma_{0,i}} = \theta_{0,i} \begin{bmatrix} -XB_i + \breve{B}_f\bar{D} \\ -ZB_i \end{bmatrix}, i = 1,2,\cdots,d,$$

$$\Psi_{\Gamma_{i,j}} = \theta_{i,j} \begin{bmatrix} X(B_i - B_j) \\ Z(B_i - B_j) \end{bmatrix}, \ 1 \le i < j \le d,$$

$$\Psi_{\bar{H}} = \begin{bmatrix} L_c - \breve{D}_f\bar{C} & L_c - \breve{D}_f\bar{C} - \breve{C}_f \end{bmatrix}, \ \Psi_{\bar{G}} = \bar{G},$$

$$\Psi_{H_0} = \beta_0[-D_f C_0 \quad -D_f C_0], \ \Psi_{G_0} = \beta_0 G_0.$$

If the LMI (2.20) is feasible, the parameters of the H_∞ filter (2.5) can be designed as

$$A_f = M^{-1}\breve{A}_f Z^{-1} N^{-T}, \quad B_f = M^{-1}\breve{B}_f, \quad C_f = \breve{C}_f Z^{-1} N^{-T} \tag{2.22}$$

where M and N are two nonsingular constant matrices satisfying

$$MN^T = I - XZ^{-1} \tag{2.23}$$

Proof Suppose that the inequality (2.20) holds, it follows readily from (2.20) that

$$\begin{bmatrix} -X & -Z \\ -Z & -Z \end{bmatrix} < 0 \tag{2.24}$$

which, by Schur complement, gives that $X - Z > 0$. Therefore, there always exist nonsingular matrices M and N such that (2.23) holds. Now, let

$$\Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} \tag{2.25}$$

$$\Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix} \tag{2.26}$$

$$Y = Z^{-1} \tag{2.27}$$

$$P = \Pi_1 \Pi_2^{-1} \tag{2.28}$$

and then

$$P = \begin{bmatrix} X & M \\ M^T & W \end{bmatrix} \tag{2.29}$$

where $W = -M^T Y N^{-T} = N^{-1} Y (X - Y^{-1}) Y N^{-T} > 0$.

It is clear that $X - M W^{-1} M^T = (I - XY)(X - Y^{-1})(I - YX) > 0$, which, by Schur complement, implies that

$$P > 0 \tag{2.30}$$

Pre- and post-multiplying the inequality of (2.20) by $\operatorname{diag}\left\{I, Y, I, \underbrace{I, Y, I, Y, \cdots, I, Y}_{2(d+2)}, \underbrace{I, Y, \cdots, I, Y}_{d(d+1)}, I, I\right\}$ leads to

$$\begin{bmatrix} -\Pi & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \bar{\Lambda}_{11} & \bar{\Lambda}_{12} & \bar{\Pi}_1 & * \\ \bar{\Lambda}_{21} & \bar{\Lambda}_{22} & 0 & \bar{\Pi}_2 \end{bmatrix} < 0 \tag{2.31}$$

where

$$\begin{gathered}
\bar{\Lambda}_{11} = \begin{bmatrix} \Lambda_{\bar{\Phi}}^T & \Lambda_{\Phi_0}^T & \Lambda_{\Phi_1}^T & \cdots & \Lambda_{\Phi_d}^T \end{bmatrix}^T, \\
\bar{\Lambda}_{12} = \begin{bmatrix} \Lambda_{\bar{\Gamma}}^T & \Lambda_{\Gamma_0}^T & \Lambda_{\Gamma_1}^T & \cdots & \Lambda_{\Gamma_d}^T \end{bmatrix}^T, \\
\bar{\Lambda}_{21} = \begin{bmatrix} \Lambda_{\Phi_{0,1}}^T & \cdots & \Lambda_{\Phi_{0,d}}^T & \Lambda_{\Phi_{1,2}}^T \cdots & \Lambda_{\Phi_{d-1,d}}^T & \Lambda_{\bar{H}}^T & \Lambda_{H_0}^T \end{bmatrix}^T, \\
\bar{\Lambda}_{22} = \begin{bmatrix} \Lambda_{\Gamma_{0,1}}^T & \cdots & \Lambda_{\Gamma_{0,d}}^T & \Lambda_{\Gamma_{1,2}}^T \cdots & \Lambda_{\Gamma_{d-1,d}}^T & \Lambda_{\bar{G}}^T & \Lambda_{G_0}^T \end{bmatrix}^T, \\
\bar{\Pi}_1 = diag(\underbrace{-\Pi, -\Pi, \cdots, -\Pi}_{d+2}), \\
\bar{\Pi}_2 = diag(\underbrace{-\Pi, -\Pi, \cdots, -\Pi}_{\frac{d(d+1)}{2}}, -I, -I).
\end{gathered} \tag{2.32}$$

with

$$\Pi = \begin{bmatrix} X & I \\ I & Y \end{bmatrix},\ \Lambda_{\bar{\Phi}} = \begin{bmatrix} X\bar{A} + MB_f\bar{C} & X\bar{A}Y + MB_f\bar{C}Y + MA_fN^T \\ \bar{A} & \bar{A}Y \end{bmatrix},$$

$$\Lambda_{\Phi_0} = \theta_0 \begin{bmatrix} MB_fC_0 & MB_fC_0Y \\ 0 & 0 \end{bmatrix}, \Lambda_{\Phi_i} = \theta_i \begin{bmatrix} XA_i & XA_iY \\ A_i & A_iY \end{bmatrix},\ i = 1,2,\cdots,d,$$

$$\Lambda_{\Phi_{0,i}} = \theta_{0,i}\begin{bmatrix} MB_fC_0 - XA_i & MB_fC_0Y - XA_iY \\ -A_i & -A_iY \end{bmatrix}, i = 1,2,\cdots,d,$$

$$\Lambda_{\Phi_{i,j}} = \theta_{i,j}\begin{bmatrix} X(A_i - A_j) & X(A_i - A_j)Y \\ A_i - A_j & (A_i - A_j)Y \end{bmatrix}, \ 1 \le i < j \le d,$$

$$\Lambda_{\bar{\Gamma}} = \begin{bmatrix} X\bar{B} + MB_f\bar{D} \\ \bar{B} \end{bmatrix}, \ \Lambda_{\Gamma_0} = \theta_0\begin{bmatrix} MB_f\bar{D} \\ 0 \end{bmatrix}, \ \Lambda_{\Gamma_i} = \theta_i\begin{bmatrix} XB_i \\ B_i \end{bmatrix}, i = 1,2,\cdots,d,$$

$$\Lambda_{\Gamma_{0,i}} = \theta_{0,i}\begin{bmatrix} -XB_i + MB_f\bar{D} \\ -B_i \end{bmatrix}, i = 1,2,\cdots,d,$$

$$\Lambda_{\Gamma_{i,j}} = \theta_{i,j}\begin{bmatrix} X(B_i - B_j) \\ B_i - B_j \end{bmatrix}, \ 1 \le i < j \le d,$$

$$\Lambda_{\bar{H}} = \begin{bmatrix} L_c - D_f\bar{C} & L_cY - D_f\bar{C}Y - C_fN^T \end{bmatrix}, \ \Lambda_{\bar{G}} = \bar{G},$$

$$\Lambda_{H_0} = \beta_0[-D_fC_0 \quad -D_fC_0Y], \ \Lambda_{G_0} = \beta_0G_0.$$

Thus, we can rewrite LMI Equation (2.20) as

$$\begin{bmatrix}
-\Pi_2^TP\Pi_2 & * & * & * & * & * & * & * & * & * & * \\
0 & -\gamma^2 I & * & * & * & * & * & * & * & * & * \\
\Pi_1^T\bar{\Phi}\Pi_2 & \Pi_1^T\bar{\Gamma} & -\breve{P} & * & * & * & * & * & * & * & * \\
\theta_0\Pi_1^T\Phi_0\Pi_2 & \theta_0\Pi_1^T\Gamma_0 & 0 & -\breve{P} & * & * & * & * & * & * & * \\
\vdots & \vdots & 0 & 0 & \ddots & * & * & * & * & * & * \\
\theta_d\Pi_1^T\Phi_d\Pi_2 & \theta_d\Pi_1^T\Gamma_d & 0 & 0 & 0 & -\breve{P} & * & * & * & * & * \\
\theta_{0,1}\Pi_1^T\Phi_{0,1}\Pi_2 & \theta_{0,1}\Pi_1^T\Gamma_{0,1} & 0 & 0 & 0 & 0 & -\breve{P} & * & * & * & * \\
\vdots & \vdots & 0 & 0 & 0 & 0 & 0 & \ddots & * & * & * \\
\theta_{d-1,d}\Pi_1^T\Phi_{d-1,d}\Pi_2 & \theta_{d-1,d}\Pi_1^T\Gamma_{d-1,d} & 0 & 0 & 0 & 0 & 0 & 0 & -\breve{P} & * & * \\
\bar{H}\Pi_2 & \bar{G} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & * \\
\beta_0H_0\Pi_2 & \beta_0G_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I
\end{bmatrix} < 0 \quad (2.33)$$

where

$$\breve{P} = \Pi_1^TP^{-1}\Pi_1$$

Pre- and post-multiplying the above inequality by $diag(\Pi_2^{-T}, I, \underbrace{-\Pi_1^{-T}, -\Pi_1^{-T}, \cdots, -\Pi_1^{-T}}_{d+2}, \underbrace{\Pi_1^{-T}, \cdots, \Pi_1^{-T}}_{\frac{d(d+1)}{2}}, I, I)$, we can obtain (2.9).

The proof is completed.

Remark 2.5 In view of Theorem 2.2, the H_∞ filtering problem for systems with multiple packet dropouts and random delays can be solved in terms of the feasibility of LMI (2.20). Note that the inequality (2.20) is not only linear with respect to matrix variables $X, Z, \breve{A}_f, \breve{B}_f, \breve{C}_f, \breve{D}_f$ but also linear with respect to scalar γ^2. So, the scalar γ^2 can be included as an optimization variable to obtain a minimum attenuation level. Thus, the filtering parameters can be readily found under the minimum attenuation level γ by solving the following convex optimization problem.

Problem 2.1 The optimal H_∞ filtering problem is

$$\min_{X>0, Z>0, \breve{A}_f, \breve{B}_f, \breve{C}_f, \breve{D}_f} \rho$$

subject to (2.20) with $\rho = \gamma^2$

Then the corresponding optimal H_∞ filtering performance level γ can be obtained by $\gamma = \sqrt{\rho}$.

Remark 2.6 Up to now, we have developed the H_∞ filtering algorithm in Theorem 2.2 based on LMI technique. It is noted that the LMI-based algorithm has a polynomial-time complexity [32]. In our main results, the time complexity is obviously dependent on the maximal delay d and the dimension n of the state variable. But it can be solved easily by MATLAB LMI toolbox and all design can be implemented offline that makes the LMI method be practicable and effective. Moreover, research on LMI optimization is a very active area in the applied math, optimization and operations research community, and substantial speed-ups can be expected in the future.

2.5 Simulation example

In this section, we take the F-404 aircraft engine system as an example to show the feasibility and applicability of the proposed algorithm. Setting the sampling time as $\mathrm{T} = 0.05\ \mathrm{s}$, we obtain the following discretized nominal system matrix A and the measurement output matrix C as [33]:

$$A = \begin{bmatrix} 0.9270 & 0 & 0.1214 \\ 0.0082 & 0.9800 & -0.0189 \\ 0.0155 & 0 & 0.8885 \end{bmatrix},\ \mathrm{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and the matrices $\mathrm{B},\ \mathrm{D}$ and L are set as

$$\mathrm{B} = \begin{bmatrix} 0.12 \\ 0.1 \\ 0.12 \end{bmatrix},\ \mathrm{D} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathrm{L} = \begin{bmatrix} 0.1 & 0.1 & -0.1 \end{bmatrix}$$

Without loss of generality, we choose $\bar{\alpha}_0 = 0.5,\ \bar{\alpha}_1 = 0.2$ and $\bar{\alpha}_2 = 0.1$. That is, the on-time received rate of a packet is 0.5, the one-step delay rate is 0.05, the two-step delay rate is 0.018, and the packet dropout rate is 0.432. The exogenous disturbance to the plant is assumed to be $w_k = 2e^{-0.1k}\sin(0.05\pi k)$. The minimum attenuation level $\gamma_{min} = 0.2218$ can be obtained by solving Problem 2.1. At the same time, we can obtain the filter parameters as:

$$A_f = \begin{bmatrix} -0.6533 & 0.3078 & 0.6643 & 0.0034 & 0.0011 & 0.0036 & 0.0004 \\ 0.2905 & -0.2257 & -0.1760 & 0.0016 & 0.0035 & 0.0004 & 0.0045 \\ 0.7288 & -0.2273 & -1.1189 & 0.0000 & -0.0040 & -0.0020 & -0.0011 \\ 0.0038 & 0.0005 & -0.0024 & -0.0003 & -0.0001 & -0.0005 & 0.0001 \\ 0.0003 & 0.0048 & -0.0014 & -0.0003 & -0.0003 & 0.0001 & -0.0007 \\ 0.0046 & -0.0002 & -0.0048 & -0.0002 & -0.0001 & -0.0004 & 0.0001 \\ 0.0005 & 0.0036 & -0.0015 & -0.0003 & -0.0002 & -0.0002 & -0.0003 \end{bmatrix},$$

$$B_f = \begin{bmatrix} -0.0085 & -0.0034 & 0.0024 & 0.0008 & 0.0006 & 0.0006 & 0.0006 \\ -0.0016 & -0.0066 & 0.0077 & 0.0001 & 0.0007 & 0.0000 & 0.0004 \end{bmatrix}^T,$$

$$C_f = [0.1124 \quad 0.0743 \quad -0.1111 \quad 0.0023 \quad -0.0213 \quad -0.0018 \quad -0.0026],$$

$$D_f = [-0.0093 \quad 0.0473].$$

The simulation result of the estimated signal z_k and its estimate $\hat{z}_k$ is shown in Figure 2.2. From Figure 2.2, it can be easily seen that the proposed algorithm is effective.

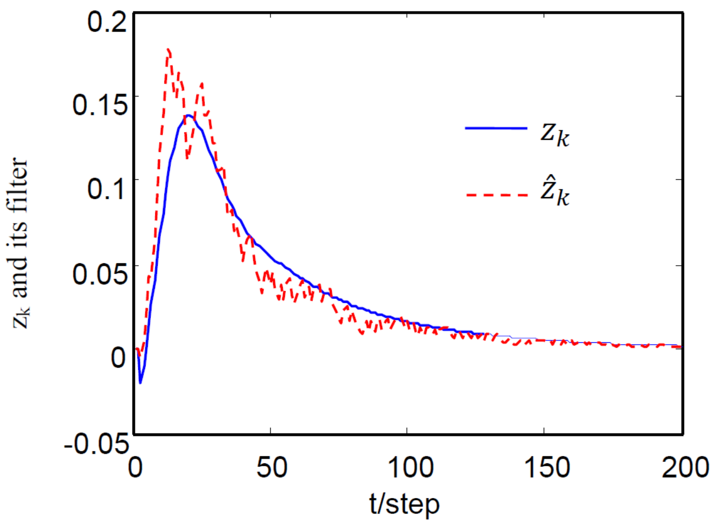

Figure 2.2 Signal z_k and its filter $\hat{z}_k$.

By calculation, we also have $\frac{\sum_{k=0}^{200} E\{\|e_k\|^2\}}{\sum_{k=0}^{200} E\{\|w_k\|^2\}} = 0.0057 < \gamma^2 = 0.0492$. It means that the H_∞ performance is satisfied, which illustrates the effectiveness of the designed filter. Furthermore, in order to show the H_∞ performance of the system for different delay rates and packet dropout rates, we fix the on-time rate of a packet as $\bar{\alpha}_0 = 0.5$, and the probability of $\bar{\alpha}_1$ and $\bar{\alpha}_2$ vary between [0,1]. That is, a packet is received by the remote filter on time is with the probability 0.5, and the one-step delay, two-step delay and the packet dropout rates are changed. The simulation result is given in Figure 2.3. From Figure 2.3, we can see that when the value of $\bar{\alpha}_2$ is fixed, and $\bar{\alpha}_1$ varies from 1 to 0, the minimum values γ_{min} become larger because the one-step delay rate is decreased and the two-step delay and the packet dropout rates are incereased. That is to say, the dynamical behavior of the filter is degraded with the larger delay and the increase of the packet dropout rates.

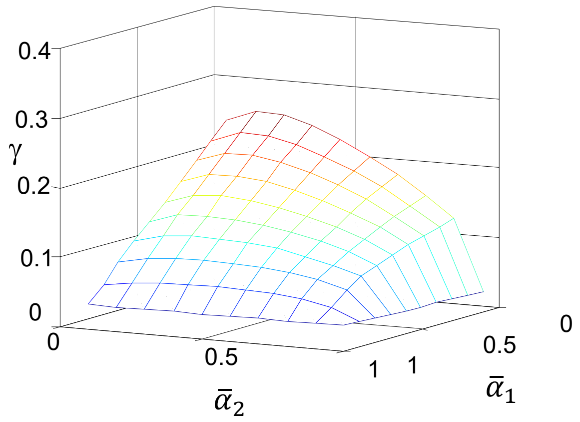

Figure 2.3 Optimal H_∞ performance for different packet-dropout rate and time-delay rate.

Moreover, we give the comparison with [29] where a full-order H_∞ filter is designed. Let $\gamma = 0.9$, we can obtain $\frac{\sum_{k=0}^{200} E\{\|e_k\|^2\}}{\sum_{k=0}^{200} E\{\|w_k\|^2\}} = 0.0431 < \gamma^2 = 0.81$ by using the algorithm in [29]. However, the value of $\frac{\sum_{k=0}^{200} E\{\|e_k\|^2\}}{\sum_{k=0}^{200} E\{\|w_k\|^2\}}$ is 0.0073 which is greatly smaller than that of reference [29]. It is shown that the filter design approach presented in this chapter can reduce the conservatism. The main reason lies in the following two aspects. One is that through the state augmentation, we can overcome the relativity of the random variables. However, due to the relativity of the random variables, the inequality scaling technique is used in the full-order filter design method of [29] which enlarges the conservatism of the design. The other is that after the state augmentation, we only choose the simple Lyapunov function when stability analysis where the Lyapunov matrix P is need to be positive definite. However, the Lyapunov function chosen in [29] have two parts including the standard Lyapunov function and another is also considered the effect of the delay where the Lyapunov matrix (P, Q_j) in every term is required to be positive definite. If we augment the state and the delayed state, the weighted matrix in the Lyapunov function is only diagonal matrix of the terms P and Q_j. The condition is restrictive than the requirement of positive definite which makes the design be conservative.

2.6 Conclusions

The optimal H_∞ filtering problem has been investigated for a class of discrete-time network-based linear systems. The network-induced uncertainties include packet dropouts and transmission delays, where the packet dropouts are possible consecutive and the delays are random but bounded. LMI-based conditions have been formulated for the existence of admissible filters, which ensure the filtering error system to be exponentially stable in the mean square sense with a prescribed H_∞ disturbance attenuation level. An F-404 aircraft engine system has been exploited to illustrate the applicability and effectiveness of the filter design methodologies proposed. Moreover, although data is transmitted via a single packet, the framework of this chapter could be applied to the case of multi-packet transmission. These topics will be studied in the future work.

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Chapter 3 H_∞ Filtering for Multiple Channel Systems with Varying Delays, Consecutive Packet Losses and Randomly Occurred Nonlinearities

3.1 Introduction

With the rapid development of computer and communication technology, networks have been increasingly used in industry due to its flexibility in modularization and economic efficiency. In most industrial process, the plant, the controller and other components are often connected over network media, where signals are transmitted through networks in the form of data packets, which is known as the so-called networked control systems (NCSs). As wide applications of NCSs in manufacturing plants, traffic, communication, aviation, and space flight, *etc*., a great deal of attention has been attracted in this area. Many results for control and filtering problems have been investigated in the literatures, see [1-7] and the references therein. Much of recent work on NCSs has been done on H_∞ design which has gained persistent attention from the early 1980s [8-12]. As for H_∞ filtering, it means that design an estimator for a given system such that the L_2 gain from the exogenous disturbance to the estimation error is less than a given level γ. In contrast with the well-known Kalman filter, the H_∞ filter does not make any assumptions on the statistics properties of the process and measurement noises which are not always available in application. So H_∞ filtering is more adaptive to the actual environment.

During the transmission through the network link from the sender to the receiver, the data packets will be lost unavoidable due to the unreliable medium and network congestion. There are several ways to deal with the data losses in network communication protocols. For example, the lost data will be required to resend in the TCP/IP which will cause more communication delay and is not acceptable for some control systems. For the real-time data transmission in NCSs, the UDP/IP is widely used because of the short communication delay. Thus, the transmission delays and packet losses are important issues in NCSs which happen randomly in nature. Besides the

Markovian modeling method [13,14], the other popular approach to describing the packet loss or transmission delay phenomenon is the Bernoulli distribution model, e.g., see [15,16] for packet loss and [17,18] for transmission delay, respectively. Different from separately considering packet losses or time delays problem, latest references have focused on dealing with the packet loss and random varying delay in a unified framework [19,20]. However, in practical engineering, the real systems are often with multiple channels. Although there are few results in considering the cases that the individual sensor has different packet missing probability [21,22] or time delay probability [23] of different channel, most of the relevant literature has been based on the hypothesis that all the sensors have the same packet loss rates or/and time delay rates. Up to now, the H_∞ filtering problem for the NCSs with multiple channels where the packet loss rates and delay rates of different sensors are simultaneously different has not been fully investigated.

On the other hand, nonlinearities are recognized to exist universally in practical systems. Filtering for nonlinear dynamical systems is an important research area. Because the Takagi-Sugeno (T-S) fuzzy model has been proven to be a conceptually simple and powerful tool to approximate complex nonlinear systems [24], many researchers have paid their attention to the fuzzy filtering method. For more details on the subject, see [25,26] and the references therein. It is worth mentioning that, in a networked environment, a number of practical systems are influenced by additive nonlinear disturbances, such as random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, *etc*., which also occur in a randomly way. The additive exogenous disturbances to the system model caused by environmental changes are named as randomly occurred nonlinearities (RONs) [27]. Take an aircraft engine system as example [28], the aircraft in air is in some way disturbed by uncontrolled external forces, such as wind gusts, gravity gradients, or structural vibrations, which may enter the systems in many different ways and can be modeled as the RONs. Among various type of nonlinearities, the sector-nonlinearity that covers Lipschitz conditions and norm-bounded conditions is quite general and can characterize the quantization and saturation functions in NCSs [29], so it has been extensively studied in the existing literatures, e.g., see [30-32] *etc*. It is well known that nonlinearity and randomness are the two main causes that have resulted in the complexity of considerable systems. To the best of our knowledge, H_∞ filtering problem for multi-channel network-based systems with varying delays, consecutive packet losses and randomly occurred nonlinearities has not been properly addressed yet despite its potential in practical applications. This motivates our current work.

The main contributions of this chapter can be summarized as follows. (1) A fairly comprehensive model is proposed to describe the measurement output transmission of the multiple channel network-based systems by introducing two diagonal matrices. For different sensors, the probabilities of the occurrence of random packet losses and transmission delays phenomena may differ from each other. The situation considered widely that all channels have the same packet loss rate or/and time delay rate is included here as a special case. (2) By employing the Lyapunov stability theory combined with the stochastic analysis approach, a linear full-order filter is designed such that, in the presence of all admissible time delays, packet losses and randomly occurred nonlinearities, the dynamics of the filtering error is guaranteed to be exponentially stable in the mean square sense, and the prescribed H_∞ disturbance rejection attenuation level is also achieved. (3) To reduce the design conservativeness, a sufficient condition of obtaining the H_∞ filter matrices is given in LMI form by introducing the slack variable to separate the Lyapunov matrices and the filter matrices which are coupled together.

3.2 Problem formulation

The considered problem is shown in Figure 3.1, in which the plant is described by the discrete-time system of the form:

$$\begin{cases} x_{k+1} = Ax_k + \sum_{i=1}^{d} \alpha_{i,k} f_i(x_k) + Dw_k \\ \tilde{y}_k = Cx_k + v_k \\ z_k = Lx_k \end{cases} \tag{3.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $\tilde{y}_k \in \mathbb{R}^r$ is the measured multi-channel output, $z_k \in \mathbb{R}^m$ is the signal to be estimated, $w_k \in \mathbb{R}^q$ and $v_k \in \mathbb{R}^r$, assumed to belong to $l_2[0,\infty)$, are the process noise and measurement noise, respectively; A, C, D and L are known constant matrices. d is the maximum number of the possibly occurred nonlinearities. The random variables $\alpha_{i,k}(1 \leq i \leq d)$ are introduced to describe the random nature of nonlinearities occurrence which are mutually uncorrelated Bernoulli distributed and have the following statistical characteristics:

$$\begin{gathered} \mathrm{Prob}\{\alpha_{i,k} = 1\} = \mathrm{E}\{\alpha_{i,k}\} = \bar{\alpha}_i, \\ \mathrm{Prob}\{\alpha_{i,k} = 0\} = 1 - \bar{\alpha}_i, \\ \mathrm{E}\{(\alpha_{i,k} - \bar{\alpha}_i)(\alpha_{j,k} - \bar{\alpha}_j)\} = \begin{cases} \bar{\alpha}_i(1 - \bar{\alpha}_i) = \vartheta_i^2, & \mathrm{i = j} \\ 0, & \mathrm{i \neq j} \end{cases}, \ (i,j = 1,2,\cdots,d) \end{gathered} \tag{3.2}$$

where $\bar{\alpha}_i \in [0,1]$ are known constants.

The nonlinear functions $f_i(x_k)$ $(i = 1,2,\cdots,d)$ are assumed to satisfy the following sector-bounded conditions with $f_i(0) = 0$, and

$$[f_i(x) - f_i(y) - U_{1i}(x-y)]^T[f_i(x) - f_i(y) - U_{2i}(x-y)] \leq 0,\ \forall \mathrm{x}, \mathrm{y} \in \mathbb{R}^n. \quad (3.3)$$

where $U_{1i}, U_{2i} \in \mathbb{R}^{n\times n}$ are known real constant matrices, and $U_{1i} - U_{2i}$ are positive definite matrices. It is customary that the nonlinear functions $f_i(x_k)$ described in (3.3) are said to belong to sectors $[U_{1i}, U_{2i}]$ [33].

Remark 3.1 It is well known that T-S fuzzy model is an effective method to analyze and synthesize nonlinear systems. But how to select the number of IF-THEN rules and the type of membership functions does not have the unified disciplines. Compared with the well-studied fuzzy filtering method, the nonlinearities existing in model (3.1) are taken as environmental disturbances which are randomly changeable with the known probabilities in terms of their sector-bounded types. Furthermore, the sector-bounded condition (3.3) can be transformed into the inequality constraints, which will be added in the subsequent LMI derivation by rigorous mathematical deduction.

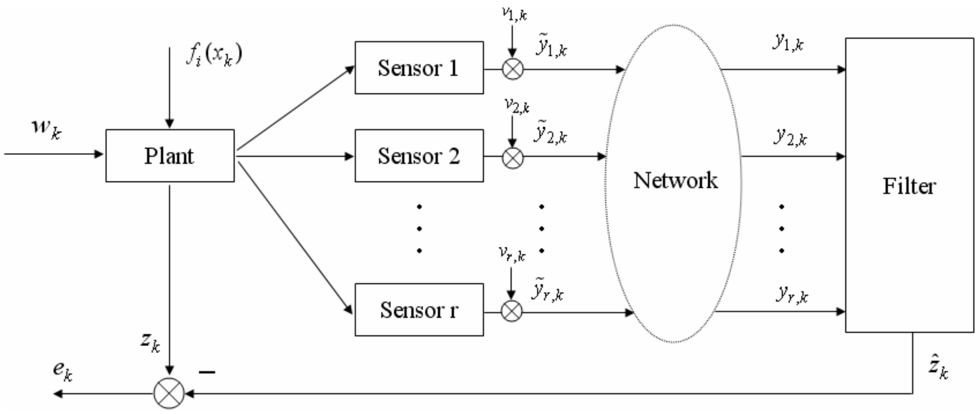

Figure 3.1 Structure of the considered networked systems.

Here, we assume that the sensors are clock-driven. The measured output $\tilde{y}_k$ is packed and sent to the remote filter side through the networks. The network protocol under consideration is UDP/IP case. The packet losses and one-step time delays will be involved during the data transmission which can be described in a unified model [19]:

$$y_k = \xi_k \tilde{y}_k + (1-\xi_k)(1-\xi_{k-1})\beta_k \tilde{y}_{k-1} + (1-\xi_k)[1-(1-\xi_{k-1})\beta_k]y_{k-1} \quad (3.4)$$

where $y_k \in \mathbb{R}^r$ is the actual signal received by the filter, and ξ_k and β_k are independent random variables satisfying Bernoulli distribution. It can be seen that the popular used model $y_k = \xi_k \tilde{y}_k + (1-\xi_k)y_{k-1}$ which describes the multiple packet dropouts and the model $y_k = \xi_k \tilde{y}_k + (1-\xi_k)\tilde{y}_{k-1}$ which describes the random one-step transmission delay are contained in model (3.4) simultaneously.

In practical engineering, however, there are a large amount of MIMO controlled systems. When the sampled data of individual sensor are transmitted through different channel link, there will be different packet loss rates and time delay rates for each channel. In considering of the investigated issues, the measurement received by the filter is described by:

$$y_k = \Xi_k \tilde{y}_k + (1-\Xi_k)(1-\Xi_{k-1})\Theta_k \tilde{y}_{k-1} + (1-\Xi_k)[1-(1-\Xi_{k-1})\Theta_k]y_{k-1} \tag{3.5}$$

where $y_k = vec_r^T\{y_{i,k}\}$, $\tilde{y}_k = vec_r^T\{\tilde{y}_{i,k}\}$, $\Xi_k = diag_r\{\xi_{i,k}\}$, $\Theta_k = diag_r\{\beta_{i,k}\}(i = 1,2,\cdots,r)$. $\xi_{i,k}$ and $\beta_{i,k}(1 \le i \le r)$ are Bernoulli distributed random variables which are uncorrelated with each other and also mutually independent of $\alpha_{i,k}(1 \le i \le d)$. The probability distributions are given by:

$$\mathrm{Prob}\{\xi_{i,k} = 1\} = E\{\xi_{i,k}\} = \bar{\xi}_i,\ \mathrm{Prob}\{\xi_{i,k} = 0\} = 1 - \bar{\xi}_i,$$
$$\mathrm{Prob}\{\beta_{i,k} = 1\} = E\{\beta_{i,k}\} = \bar{\beta}_i,\ \mathrm{Prob}\{\beta_{i,k} = 0\} = 1 - \bar{\beta}_i.$$

where $\bar{\xi}, \bar{\beta}_i \in [0,1]$ are known constants. For each channel, the corresponding probability of the on-time arrival rate, one-step delay rate, and the packet loss rate are given by $\mathrm{Prob}\{\xi_{i,k} = 1\} = \bar{\xi}_i$, $\mathrm{Prob}\{\xi_{i,k} = 0, \xi_{i,k-1} = 0, \beta_{i,k} = 1\} = (1-\bar{\xi}_i)^2\bar{\beta}_i$, and $\mathrm{Prob}\{\xi_{i,k} = 0, \xi_{i,k-1} = 1\} + \mathrm{Prob}\{\xi_{i,k} = 0, \xi_{i,k-1} = 0, \beta_{i,k} = 0\} = (1-\bar{\xi}_i)\bar{\xi}_i(1-\bar{\xi}_i)^2(1-\bar{\beta}_i)$, respectively.

Remark 3.2 It is noted from model (3.5) combined with (3.1) that the measurement y_k contains the consecutive packet losses and the one-step varying delays as well as randomly occurred nonlinearities. Recently, some results on the H_∞ filtering for NCSs with RONs have been reported in the literatures [20,27,32]. However, the phenomenon of packet dropouts is only considered in [27,32], and the network-induced time delays are not included in the presented model. In [20], the mixed random delays and packet dropouts are both investigated, but only one type of nonlinear function happens randomly. On the other hand, the data missing probability or time delay probability of different channels are the same in the above results. When the measurement data are transmitted through multiple sensors, it would be more reasonable to assume that the associated packet loss rates and time delay rates may differ from each other. Hence, the addressed filtering problem in this chapter is more complicated and reasonable.

Remark 3.3 In practice, the network-induced delay of some type of networks is less than one sample period, such as CAN, which can be described by our model. Moreover, we can obtain the on-time rate, one-step delay rate and packet dropout rate by using time-stamp technique through statistics method, from which the values of $\bar{\xi}_i$ and $\bar{\beta}_i$ can be calculated.

Now, we are interested in obtaining the estimate of the signal z_k from the measurement $y_k \in \mathbb{R}^r$ received by the filter. The full-order filter to be designed is of the following form:

$$\begin{cases} \hat{x}_{k+1} = A_f \hat{x}_k + B_f y_k \\ \hat{z}_k = L_f \hat{x}_k \end{cases} \tag{3.6}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the estimate state, and $\hat{z}_k \in \mathbb{R}^m$ is the estimated signal. A_f, B_f and L_f are the filter parameters to be designed with proper dimensions.

Note that there are two stochastic diagonal matrices in model (3.5), we will apply the stochastic analysis method to deal with the filter design. Before proceeding further, we first transform model (3.5) into a compact form. Define the new variables

$$\delta_{i,k} = (1 - \xi_{i,k})\beta_{i,k+1} \tag{3.7}$$

Then, we have

$$\Delta_k = (I - \Xi_k)\Theta_{k+1} \tag{3.8}$$

and

$$Y_k = \Delta_k \tilde{y}_k + (I - \Delta_k) y_k \tag{3.9}$$

From the fact that $\Xi_k \Delta_k = \Xi_k (I - \Xi_k)\Theta_{k+1} = 0$, it yields

$$y_k = \Xi_k \tilde{y}_k + (I - \Xi_k) Y_{k-1} \tag{3.10}$$

and

$$Y_k = (\Xi_k + \Delta_k)\tilde{y}_k + (I - \Xi_k - \Delta_k) Y_{k-1} \tag{3.11}$$

Further, the following statistical characteristics can easily be obtained as

$$\mathrm{E}\{\delta_{i,k}\} = \bar{\delta}_i = (1 - \bar{\xi}_i)\bar{\beta}_i, \tag{3.12}$$

$$\mathrm{E}\{(\xi_{i,k} - \bar{\xi}_i)(\xi_{j,k} - \bar{\xi}_j)\} = \begin{cases} \bar{\xi}_i(1 - \bar{\xi}_i) = \sigma_i^2, & \mathrm{i = j} \\ 0, & \mathrm{i \neq j} \end{cases} \tag{3.13}$$

$$\mathrm{E}\{(\delta_{i,k} - \bar{\delta}_i)(\delta_{j,k} - \bar{\delta}_j)\} = \begin{cases} \bar{\delta}_i(1 - \bar{\delta}_i) = \upsilon_i^2, & \mathrm{i = j} \\ 0, & \mathrm{i \neq j} \end{cases} \tag{3.14}$$

$$\mathrm{E}\{(\xi_{i,k} - \bar{\xi}_i)(\delta_{j,k} - \bar{\delta}_j)\} = \begin{cases} -\bar{\xi}_i \bar{\delta}_i = -\rho_i^2, & \mathrm{i = j} \\ 0, & \mathrm{i \neq j} \end{cases} \tag{3.15}$$

Letting the estimation error be $e_k = z_k - \hat{z}_k$, the error dynamics can be obtained from (3.1), (3.6), (3.10) and (3.11) as:

$$\begin{cases}\eta_{k+1} = \tilde{A}_k\eta_k + \sum_{i=1}^{d}\bar{\alpha}_i H f_i(x_k) + \sum_{i=1}^{d}(\alpha_{i,k} - \bar{\alpha}_i) H f_i(x_k) + \bar{\Gamma}_w w_k + \tilde{\Gamma}_k v_k \\ e_k = \bar{L}\eta_k \end{cases} \tag{3.16}$$

where

$$\eta_k = [x_k^T \quad \hat{x}_k^T \quad Y_{k-1}^T]^T,\ H = [I \quad 0 \quad 0]^T,\ \bar{\Gamma}_w = [D^T \quad 0 \quad 0]^T,\ \bar{L} = [L \quad -L_f \quad 0],$$

$$\tilde{A}_k = \bar{A} + \breve{A}_1 + \breve{A}_2,\ \ \tilde{\Gamma}_k = \bar{\Gamma}_v + \breve{\Gamma}_1 + \breve{\Gamma}_2,$$

$$\bar{A} = \begin{bmatrix} A & 0 & 0 \\ B_f\bar{\Xi}C & A_f & B_f(I-\bar{\Xi}) \\ (\bar{\Xi}+\bar{\Delta})C & 0 & I-\bar{\Xi}-\bar{\Delta}\end{bmatrix},\ \bar{\Gamma}_v = \begin{bmatrix} 0 \\ B_f\bar{\Xi} \\ \bar{\Xi}+\bar{\Delta}\end{bmatrix},$$

$$\breve{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ B_f(\Xi_k-\bar{\Xi})C & 0 & -B_f(\Xi_k-\bar{\Xi}) \\ (\Xi_k-\bar{\Xi})C & 0 & -(\Xi_k-\bar{\Xi})\end{bmatrix} = \sum_{j=1}^{r}(\xi_{j,k}-\bar{\xi}_j)\bar{C}_{1,j}$$

$$\breve{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\Delta_k-\bar{\Delta})C & 0 & -(\Delta_k-\bar{\Delta})\end{bmatrix} = \sum_{j=1}^{r}(\delta_{j,k}-\bar{\delta}_j)\bar{C}_{2,j}$$

$$\breve{\Gamma}_1 = \begin{bmatrix} 0 \\ B_f(\Xi_k-\bar{\Xi}) \\ \Xi_k-\bar{\Xi}\end{bmatrix} = \sum_{j=1}^{r}(\xi_{j,k}-\bar{\xi}_j)\bar{B}_{1,j}$$

$$\breve{\Gamma}_2 = \begin{bmatrix} 0 \\ 0 \\ \Delta_k-\bar{\Delta}\end{bmatrix} = \sum_{j=1}^{r}(\delta_{j,k}-\bar{\delta}_j)\bar{B}_{2,j}$$

$$\bar{C}_{1,j} = \begin{bmatrix} 0 & 0 & 0 \\ B_fC_j & 0 & -B_fI_j \\ C_j & 0 & -I_j\end{bmatrix},\ \bar{C}_{2,j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_j & 0 & -I_j\end{bmatrix},\ \bar{B}_{1,j} = \begin{bmatrix} 0 \\ B_fI_j \\ I_j\end{bmatrix},\ \bar{B}_{2,j} = \begin{bmatrix} 0 \\ 0 \\ I_j\end{bmatrix}.$$

$$C_j = diag\{\underbrace{0,\cdots,0}_{j-1},1,\underbrace{0,\cdots,0}_{r-j}\}C$$

$$I_j = diag\{\underbrace{0,\cdots,0}_{j-1},1,\underbrace{0,\cdots,0}_{r-j}\},\ j = 1,2,\cdots,r.$$

and $\bar{\Xi} = \mathrm{E}\{\Xi_k\}$, $\bar{\Delta} = \mathrm{E}\{\Delta_k\}$, where $\xi_{i,k}$ and $\delta_{i,k}$ are replaced by their expectations $\bar{\xi}_i$ and $\bar{\delta}_i$, $i = 1,2,\cdots,r$.

Remark 3.4 Here, we give a new variable definition method in (3.7) compared with [19]. If we use the augmentation method in [19], the dimension of η_k will be $2n+3r$. However, the method we proposed here results in the dimension of augmented state η_k to be $2n+r$. Moreover, the considered system is with multiple channels, *i.e.*, $r > 1$. It is obvious that the proposed method can greatly reduce the computational burden.

Definition 3.1 For a given scalar $\gamma > 0$, the filtering error system (3.16) is said to be exponentially stable in the mean square with an H_∞ performance level γ if the following two performance requirements hold:

(i) The filtering error system (3.16) is said to be exponentially stable in the mean square under $w_k = 0$ and $v_k = 0$, if there exist constants $\varphi > 0$ and $\tau \in (0,1)$, such that

$$E\{\|\eta_k\|^2\} \leq \varphi\tau^k E\{\|\eta_0\|^2\}, \text{ for all } \eta_0 \in \mathbb{R}^{2n+r}, k \in I^+.$$

(ii) The filtering error system (3.16) is said to be satisfied with the H_∞ performance constraint if under zero initial condition and for any non-zero $w_k \in l_2[0,\infty)$ and $v_k \in l_2[0,\infty)$, the filtering error e_k satisfies

$$\sum_{k=0}^{\infty} E\{\|e_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\|w_k\|^2 + \|v_k\|^2\}. \tag{3.17}$$

In this chapter, we aim to design a linear full-order filter of the form (3.6) for the system (3.1) such that, for all admissible varying one-step delays, sensor measurement losses, multiple randomly occurred nonlinearities and exogenous disturbances, the filtering error system (3.16) is exponentially stable in the mean square with an H_∞ performance level $\gamma > 0$.

3.3 Filtering performance analysis

In this section, we present the main results of exponential stability in the mean square and H_∞ performance analysis. The following Lemmas are useful for the derivation of our main results.

Lemma 3.1 [34] Let $V(\eta_k)$ be a Lyapunov functional. If there exist real scalars $\lambda \geq 0$, $\mu > 0$, $\nu > 0$, and $0 < \psi < 1$, such that

$$\mu\|\eta_k\|^2 \leq V(\eta_k) \leq \nu\|\eta_k\|^2$$
$$E\{V(\eta_{k+1})|\eta_k\} - V(\eta_k) \leq \lambda - \psi V(\eta_k)$$

then the sequence η_k satisfies

$$E\{\|\eta_k\|^2\} \leq \frac{\nu}{\mu}\|\eta_0\|^2(1-\psi)^k + \frac{\lambda}{\mu\psi}$$

Lemma 3.2 [35] (S-Procedure) Let $W_0(x), W_1(x), \dots, W_p(x)$ be quadratic functions of $x \in \mathbb{R}^n$, *i.e.*, $W_i(x) = x^T T_i x$, $i = 0,1,\cdots,p$, with $T_i = T_i^T$. Then the implication $W_1(x) \leq 0, \dots, W_p(x) \leq 0 \Rightarrow W_0(x) \leq 0$ holds if there exist scalars $\tau_1, \tau_2, \dots, \tau_p > 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i \leq 0$$

Lemma 3.3 [36] For matrices A, $P_0 > 0$ and $P_1 > 0$, the following inequality:

$$A^T P_1 A - P_0 < 0,$$

is equivalent to that there exists a matrix W such that

$$\begin{bmatrix} -P_0 & * \\ WA & P_1 - W - W^T \end{bmatrix} < 0$$

We first establish the exponential stability in the mean square for the filtering error system (3.16) under $w_k = 0$ and $v_k = 0$.

Theorem 3.1 Consider the filtering error system (3.16) with given filter parameters A_f, B_f and L_f. Then, the system (3.16) is exponentially stable in the mean square under $w_k = 0$ and $v_k = 0$. If there exist a positive definite matrix $0 < \mathrm{P} \in \mathbb{R}^{(2n+r)\times(2n+r)}$, and constant scalars $\lambda_1, \lambda_2, \dots, \lambda_d > 0$ such that the following matrix inequality holds

$$\Pi = \begin{bmatrix} \Pi_{11} & * \\ \Pi_{21} & \Pi_{22} \end{bmatrix} < 0 \tag{3.18}$$

where

$$\Pi_{11} = -P + \bar{A}^T P \bar{A} + \sum_{j=1}^{r} \sigma_j^2 \bar{C}_{1,j}^T P \bar{C}_{1,j} + \sum_{j=1}^{r} v_j^2 \bar{C}_{2,j}^T P \bar{C}_{2,j} - \sum_{j=1}^{r} \rho_j^2 \bar{C}_{1,j}^T P \bar{C}_{2,j} - \sum_{j=1}^{r} \rho_j^2 \bar{C}_{2,j}^T P \bar{C}_{1,j} - \sum_{l=1}^{d} \lambda_l^2 H \widetilde{U}_{1l} H^T,$$

$$\Pi_{21} = \bar{G}^T P \bar{A} - U,\ \Pi_{22} = \bar{G}^T P \bar{G} + G^T \breve{P} G - \Lambda,$$

$$\bar{G} = vec_d\{\bar{\alpha}_i H\},\ G = diag_d\{\vartheta_i H\},\ \breve{P} = diag_d\{P\},$$

$$\Lambda = diag_d\{\lambda_i I\},\ \breve{U} = vec_d^T\{\lambda_i H \widetilde{U}_{2i}\}$$

$$\widetilde{U}_{1l} = (U_{1l}^T U_{2l} + U_{2l}^T U_{1l})/2,\ \widetilde{U}_{2l} = -(U_{1l}^T + U_{2l}^T)/2,\ l = 1,2,\cdots,d$$

with $\bar{\alpha}_i$, ϑ_i, $i = 1,2,\cdots,d$ are defined by (3.2), and σ_j, v_j, ρ_j, $j = 1,2,\cdots,r$ are defined by (3.13)-(3.15).

Proof Select the Lyapunov functional candidate as

$$V_k(\eta_k) = \eta_k^T P \eta_k \tag{3.19}$$

When $w_k = 0$ and $v_k = 0$, calculating the difference V_k along the trajectory of system (3.16) and taking the mathematical expectation, we have

$$E\{\Delta V_k\} = E\{V_{k+1}(\eta_{k+1})|\eta_k\} - V_k(\eta_k)$$

$$= E\left\{\left[\tilde{A}_k \eta_k + \sum_{i=1}^{d} \bar{\alpha}_i H f_i(x_k) + \sum_{i=1}^{d} (\alpha_{i,k} - \bar{\alpha}_i) H f_i(x_k)\right]^T P \left[\tilde{A}_k \eta_k + \sum_{i=1}^{d} \bar{\alpha}_i H f_i(x_k) + \sum_{i=1}^{d} (\alpha_{i,k} - \bar{\alpha}_i) H f_i(x_k)\right] \Big| \eta_k\right\} - \eta_k^T P \eta_k$$

$$= \eta_k^T E\{(\bar{A}^T P \bar{A} + \breve{A}_1^T P \breve{A}_1 + \breve{A}_2^T P \breve{A}_2 + \breve{A}_1^T P \breve{A}_2 + \breve{A}_2^T P \breve{A}_1 - P)\}\eta_k + \sum_{i=1}^{d} \vartheta_i^2 f_i^T(x_k) H^T P H f_i(x_k)$$

$$+ \sum_{i=1}^{d} \bar{\alpha}_i \bar{\alpha}_j f_i^T(x_k) H^T P H f_j(x_k) + \sum_{i=1}^{d} \bar{\alpha}_i f_i^T(x_k) H^T P \bar{A} \eta_k + \sum_{i=1}^{d} \bar{\alpha}_i \eta_k^T \bar{A}^T P H f_i(x_k)$$

From the definitions of $\breve{A}_1$ and $\breve{A}_2$ in (3.16), we have

$$E\{\breve{A}_1^T P \breve{A}_1\} = \sum_{j=1}^{r} \sigma_j^2 \bar{C}_{1,j}^T P \bar{C}_{1,j},\ E\{\breve{A}_2^T P \breve{A}_2\} = \sum_{j=1}^{r} v_j^2 \bar{C}_{2,j}^T P \bar{C}_{2,j},$$

$$E\{\breve{A}_1^T P \breve{A}_2\} = -\sum_{j=1}^{r} \rho_j^2 \bar{C}_{1,j}^T P \bar{C}_{2,j},\ \mathrm{E}\{\breve{A}_2^T P \breve{A}_1\} = -\sum_{j=1}^{r} \rho_j^2 \bar{C}_{2,j}^T P \bar{C}_{1,j}$$

Then, one can obtain that

$$E\{\Delta V_k\} = E\{V_{k+1}(\eta_{k+1})|\eta_k\} - V_k(\eta_k) = \zeta_k^T \Omega \zeta_k \tag{3.20}$$

where $\zeta_k = [\eta_k^T \quad f^T(x_k)]^T$ with $f(x_k) = vec_d^T\{f_i(x_k)\}$, and Ω has the structure of:

$$\Omega = \begin{bmatrix} \Omega_{11} & * \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \tag{3.21}$$

with

$$\Omega_{11} = -P + \bar{A}^T P \bar{A} + \sum_{j=1}^{r} \sigma_j^2 \bar{C}_{1,j}^T P \bar{C}_{1,j} + \sum_{j=1}^{r} v_j^2 \bar{C}_{2,j}^T P \bar{C}_{2,j} - \sum_{j=1}^{r} \rho_j^2 \bar{C}_{1,j}^T P \bar{C}_{2,j} - \sum_{j=1}^{r} \rho_j^2 \bar{C}_{2,j}^T P \bar{C}_{1,j},$$

$$\Omega_{21} = \bar{G}^T P\bar{A},\ \Omega_{22} = \bar{G}^T P\bar{G} + G^T \breve{P} G,$$

Note that from (3.3), it follows readily that:

$$[f_l(H^T\eta_k) - U_{1l}H^T\eta_k]^T[f_l(H^T\eta_k) - U_{2l}H^T\eta_k] \leq 0$$

Thus, one can obtain

$$\begin{bmatrix} \eta_k \\ f_l(x_k) \end{bmatrix}^T \begin{bmatrix} H\tilde{U}_{1l}H^T & H\tilde{U}_{2l} \\ \tilde{U}_{2l}^T H^T & I \end{bmatrix} \begin{bmatrix} \eta_k \\ f_l(x_k) \end{bmatrix},\ l = 1,2,\cdots,d \tag{3.22}$$

where $\tilde{U}_{1l}$ and $\tilde{U}_{2l}$ are defined by Theorem 3.1.

According to Lemma 3.2, we can derive that $E\{\Delta V_k\} \leq \zeta_k^T\Omega\zeta_k < 0$ with constraints (3.22) holds if there exist scalars $\lambda_1,\cdots,\lambda_d > 0$ such that

$$E\{\Delta V_k\} \leq \zeta_k^T\Omega\zeta_k - \sum_{l=1}^{d}\lambda_l \begin{bmatrix} \eta_k \\ f_l(x_k) \end{bmatrix}^T \begin{bmatrix} H\tilde{U}_{1l}H^T & H\tilde{U}_{2l} \\ \tilde{U}_{2l}^T H^T & I \end{bmatrix} \begin{bmatrix} \eta_k \\ f_l(x_k) \end{bmatrix} = \zeta_k^T\Pi\zeta_k < 0 \tag{3.23}$$

Π is defined by (3.18). And (3.18) given in Theorem 3.1 implies $\Pi < 0$. Moreover,

$$E\{\Delta V_k\} \leq \zeta_k^T\Pi\zeta_k - \lambda_{min}(-\Pi)\zeta_k^T\zeta_k < -\phi\zeta_k^T\zeta_k \tag{3.24}$$

where $0 < \phi < \min\{\lambda_{min}(-\Pi), \lambda_{max}(P)\}$.

From (3.24), we have $\mathrm{E}\{\Delta V_k\} < \frac{\phi}{\kappa}V_k = -\psi V_k$, where $\kappa = \lambda_{max}(P)$. By Definition 3.1 and Lemma 3.1, it can be verified that the filtering error system (3.16) is exponentially mean-square stable. The proof is completed.

Next, we will analyze the H_∞ performance of the filtering error system (3.16) when $w_k \neq 0$ and $v_k \neq 0$.

Theorem 3.2 Suppose that the filter parameters A_f, B_f and L_f are given. The filtering error system (3.16) is exponentially mean-square stable and when $w_k \neq 0$ and $v_k \neq 0$, the H_∞ norm constraint (3.17) is achieved for a given scalar $\gamma > 0$ if there exist a positive definite matrix $0 < P \in \mathbb{R}^{(2n+r)\times(2n+r)}$ and scalars $\lambda_1,\cdots,\lambda_d > 0$ such that the following matrix inequality holds

$$\bar{\Pi} = \begin{bmatrix} -\bar{P} & * & * & * & * & * & * & * & * & * \\ -\breve{U} & -\Lambda & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ 0 & 0 & 0 & -\gamma^2 I & * & * & * & * & * & * \\ \breve{P}\breve{C}_1 & 0 & 0 & \breve{P}\breve{B}_1 & -\breve{P} & * & * & * & * & * \\ \breve{P}\breve{C}_2 & 0 & 0 & \breve{P}\breve{B}_2 & 0 & -\breve{P} & * & * & * & * \\ \breve{P}\breve{C}_{12} & 0 & 0 & \breve{P}\breve{B}_{12} & 0 & 0 & -\breve{P} & * & * & * \\ P\bar{A} & P\bar{G} & P\bar{\Gamma}_w & P\bar{\Gamma}_v & 0 & 0 & 0 & -P & * & * \\ 0 & \hat{P}G & 0 & 0 & 0 & 0 & 0 & 0 & -\breve{P} & * \\ \bar{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{3.25}$$

where

$$\bar{\sigma}_i^2 = \sigma_i^2 - \rho_i^2 = \bar{\xi}_i(1-\bar{\xi}_i)(1-\bar{\beta}_i),\ i = 1,2,\cdots,r,$$

$$\bar{v}_i^2 = v_i^2 - \rho_i^2 = \bar{\beta}_i\big(1-\bar{\xi}_i\big)^2(1-\bar{\beta}_i),\ i = 1,2,\cdots,r,$$

$$\breve{C}_1 = vec_r^T\{\bar{\sigma}_i\bar{C}_{1,i}^T\},\ \breve{C}_2 = vec_r^T\{\bar{\upsilon}_i\bar{C}_{2,i}^T\},\ \breve{C}_{12} = vec_r^T\{\bar{\rho}_i(\bar{C}_{1,i} - \bar{C}_{2,i})^T\},$$
$$\breve{B}_1 = vec_r^T\{\bar{\sigma}_i\bar{B}_{1,i}^T\},\ \breve{B}_2 = vec_r^T\{\bar{\upsilon}_i\bar{B}_{2,i}^T\},\ \breve{B}_{12} = vec_r^T\{\bar{\rho}_i(\bar{B}_{1,i} - \bar{B}_{2,i})^T\},$$
$$\breve{P} = diag_r\{P\},\ \bar{P} = P + \textstyle\sum_{l=1}^{d}\lambda_l H\widetilde{U}_{1l}H^T,$$

and Λ, $\breve{P}$, $\bar{G}$, and G are defined by Theorem 3.1.

Proof Now let us deal with the H_∞ performance of the system (3.16) when $w_k \neq 0$ and $v_k \neq 0$. Consider the same Lyapunov functional as in Theorem 3.1. Note that the following equations hold

$$\begin{gathered}
E\{\breve{\Gamma}_1^T P\breve{\Gamma}_1\}=\textstyle\sum_{j=1}^{r}\sigma_j^2\bar{B}_{1,j}^T P\bar{B}_{1,j},\ \mathrm{E}\{\breve{\Gamma}_1^T P\breve{\Gamma}_2\} = -\sum_{j=1}^{r}\rho_j^2\bar{B}_{1,j}^T P\bar{B}_{2,j},\\
E\{\breve{\Gamma}_2^T P\breve{\Gamma}_2\}=\textstyle\sum_{j=1}^{r}\upsilon_j^2\bar{B}_{2,j}^T P\bar{B}_{2,j},\ \mathrm{E}\{\breve{\Gamma}_2^T P\breve{\Gamma}_1\} = -\sum_{j=1}^{r}\rho_j^2\bar{B}_{2,j}^T P\bar{B}_{1,j},\\
E\{\breve{\Gamma}_1^T P\breve{\mathrm{A}}_1\}=\textstyle\sum_{j=1}^{r}\sigma_j^2\bar{B}_{1,j}^T P\bar{C}_{1,j},\ \mathrm{E}\{\breve{\Gamma}_1^T P\breve{\mathrm{A}}_2\} = -\sum_{j=1}^{r}\rho_j^2\bar{B}_{1,j}^T P\bar{C}_{2,j},\\
E\{\breve{\Gamma}_2^T P\breve{\mathrm{A}}_2\}=\textstyle\sum_{j=1}^{r}\upsilon_j^2\bar{B}_{2,j}^T P\bar{C}_{2,j},\ \mathrm{E}\{\breve{\Gamma}_2^T P\breve{\mathrm{A}}_1\} = -\sum_{j=1}^{r}\rho_j^2\bar{B}_{2,j}^T P\bar{C}_{1,j},
\end{gathered} \tag{3.26}$$

Then, by a similar calculation as in the proof of Theorem 3.1, we have

$$E\{V_{k+1}(\eta_{k+1})|\eta_k\} - V_k(\eta_k) + E\{e_k^T e_k\} - \gamma^2 E\{w_k^T w_k + v_k^T v_k\} = \varsigma_k^T\bar{\Omega}\varsigma_k \tag{3.27}$$

where

$$\varsigma_k = [\zeta_k^T \quad w_k^T \quad v_k^T]^T,$$
$$\bar{\Omega} = \begin{bmatrix} \bar{\Omega}_{11} & * & * & * \\ \bar{\Omega}_{21} & \bar{\Omega}_{22} & * & * \\ \bar{\Omega}_{31} & \bar{\Omega}_{32} & \bar{\Omega}_{33} & * \\ \bar{\Omega}_{41} & \bar{\Omega}_{42} & \bar{\Omega}_{43} & \bar{\Omega}_{44} \end{bmatrix},$$
$$\bar{\Omega}_{11} = \Pi_{11} + \bar{L}^T\bar{L},\ \bar{\Omega}_{21} = \Pi_{21},\ \bar{\Omega}_{22} = \Pi_{22},$$
$$\bar{\Omega}_{31} = \bar{\Gamma}_w^T P\bar{A},\ \bar{\Omega}_{32} = \bar{\Gamma}_w^T P\bar{G},\ \bar{\Omega}_{33} = \bar{\Gamma}_w^T P\bar{\Gamma}_w - \gamma^2 I,$$
$$\bar{\Omega}_{41} = \bar{\Gamma}_v^T P\bar{A} + \sum_{j=1}^{r}\sigma_j^2\bar{B}_{1,j}^T P\bar{C}_{1,j} + \sum_{j=1}^{r}\upsilon_j^2\bar{B}_{2,j}^T P\bar{C}_{2,j} - \sum_{j=1}^{r}\rho_j^2\bar{B}_{1,j}^T P\bar{C}_{2,j} - \sum_{j=1}^{r}\rho_j^2\bar{B}_{2,j}^T P\bar{C}_{1,j}$$
$$\bar{\Omega}_{42} = \bar{\Gamma}_v^T P\bar{G},\ \bar{\Omega}_{43} = \bar{\Gamma}_v^T P\bar{\Gamma}_w,$$
$$\bar{\Omega}_{44} = \bar{\Gamma}_v^T P\bar{\Gamma}_v + \sum_{j=1}^{r}\sigma_j^2\bar{B}_{1,j}^T P\bar{B}_{1,j} + \sum_{j=1}^{r}\upsilon_j^2\bar{B}_{2,j}^T P\bar{B}_{2,j} - \sum_{j=1}^{r}\rho_j^2\bar{B}_{1,j}^T P\bar{B}_{2,j} - \sum_{j=1}^{r}\rho_j^2\bar{B}_{2,j}^T P\bar{B}_{1,j} - \gamma^2 I$$

where Π_{11}, Π_{21} and Π_{22} are defined by Theorem 3.1.

It is worth mentioning that there exist the terms $-\bar{C}_{1,j}^T P\bar{C}_{2,j} - \bar{C}_{2,j}^T P\bar{C}_{1,j}$, $j = 1,2,\cdots,r$ in the expression of Π_{11}, and for the matrices $\bar{C}_{1,j}$ and $\bar{C}_{2,j}$, we have

$$-\bar{C}_{1,j}^T P\bar{C}_{2,j} - \bar{C}_{2,j}^T P\bar{C}_{1,j} = (\bar{C}_{1,j} - \bar{C}_{2,j})^T P(\bar{C}_{1,j} - \bar{C}_{2,j}) - \bar{C}_{1,j}^T P\bar{C}_{1,j} - \bar{C}_{2,j}^T P\bar{C}_{2,j} \tag{3.28}$$

Performing the same relation on the terms of $-\bar{B}_{1,j}^T P\bar{B}_{2,j} - \bar{B}_{2,j}^T P\bar{B}_{1,j}$ and $-\bar{B}_{1,j}^T P\bar{C}_{2,j} - \bar{B}_{2,j}^T P\bar{C}_{1,j}$ in (3.27), we have

$$-\bar{B}_{1,j}^T P\bar{B}_{2,j} - \bar{B}_{2,j}^T P\bar{B}_{1,j} = (\bar{B}_{1,j} - \bar{B}_{2,j})^T P(\bar{B}_{1,j} - \bar{B}_{2,j}) - \bar{B}_{1,j}^T P\bar{B}_{1,j} - \bar{B}_{2,j}^T P\bar{B}_{2,j} \tag{3.29}$$

and

$$-\bar{B}_{1,j}^T P\bar{C}_{2,j} - \bar{B}_{2,j}^T P\bar{C}_{1,j} = (\bar{B}_{1,j} - \bar{B}_{2,j})^T P(\bar{C}_{1,j} - \bar{C}_{2,j}) - \bar{B}_{1,j}^T P\bar{B}_{1,j} - \bar{C}_{2,j}^T P\bar{C}_{2,j} \tag{3.30}$$

Substituting (3.28)-(3.30) into (3.27), it yields from Schur complement that $\overline{\Pi} < 0$ implies $\overline{\Omega} < 0$, that is

$$E\{\Delta V_k\} + E\{e_k^T e_k\} - \gamma^2 E\{w_k^T w_k + v_k^T v_k\} = \varsigma_k^T \overline{\Omega} \varsigma_k < 0 \tag{3.31}$$

Summing up (3.31) from 0 to ∞ with respect to k and by the zero initial condition, the following is obtained

$$\sum_{k=0}^{\infty} E\{\|e_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\|w_k\|^2 + \|v_k\|^2\}$$

Thus the H_∞ performance holds**.**

Moreover, it is clear that $\overline{\Omega} < 0$ implies $\Pi < 0$. According to Theorem 3.1, the filtering error system (3.16) is exponentially stable in the mean square. This completes the proof.

3.4 H_∞ filter design

Theorem 3.2 presents a sufficient condition for the exponentially stable in the mean square and the H_∞ performance analysis of the filtering error system (3.16), which is an LMI condition when the filter parameters are given. But our aim is to design the filter parameters. In this case, Theorem 3.2 cannot be used for filter design directly because the condition is not an LMI if the filter parameters are taken as the matrix variables. We are now interested in determining the filter matrices in (3.6) by transforming the condition obtained in Theorem 3.2 into an LMI constraint. At this stage, the slack variable will be introduced to separate the Lyapunov matrices and the filter matrices which are coupled together in Theorem 3.2. Moreover, it can lead to less conservative results.

Theorem 3.3 For the system (3.1) with varying communication delays, consecutive packet losses and randomly occurred nonlinearities, there exists an H_∞ filter of the form (3.6) such that the filtering error system (3.16) is exponentially stable in the mean square under $w_k = 0$ and $v_k = 0$ and also satisfies condition (3.17) under zero initial condition for any nonzero $w_k \in l_2[0, \infty)$ and $v_k \in l_2[0, \infty)$, if there exist matrices $\tilde{P} = \tilde{P}^T > 0$, $\mathrm{V} = \begin{bmatrix} V_1 & V_2 & V_3 \\ V_4 & V_2 & V_5 \\ V_6 & 0 & V_7 \end{bmatrix}$, $\tilde{A}_f$, $\tilde{B}_f$, $\tilde{L}_f$ and positive constant scalars $\lambda_1, \cdots, \lambda_d > 0$ such that the following LMI holds:

$$\overline{\Psi} = \begin{bmatrix} \overline{\Psi}_1 & * & * \\ \overline{\Psi}_2 & \overline{\Psi}_3 & * \\ \overline{\Psi}_4 & 0 & \overline{\Psi}_5 \end{bmatrix} \tag{3.32}$$

where

$$\overline{\Psi}_1 = \begin{bmatrix} -\tilde{P} + \sum_{l=1}^{d} \lambda_l H \breve{U}_{1l} H^T & * & * & * \\ -\breve{U} & -\Lambda & * & * \\ 0 & 0 & -\gamma^2 I & * \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},$$

$$\overline{\Psi} = \begin{bmatrix} \overline{\Upsilon}_1 & 0 & 0 & \overline{\Upsilon}_4 \\ \overline{\Upsilon}_2 & 0 & 0 & \overline{\Upsilon}_5 \\ \overline{\Upsilon}_3 & 0 & 0 & \overline{\Upsilon}_6 \end{bmatrix},\ \overline{\Upsilon}_1 = \begin{bmatrix} \overline{\Upsilon}_{l,1}^T & \overline{\Upsilon}_{l,2}^T & \dots & \overline{\Upsilon}_{l,r}^T \end{bmatrix}^T, l = 1,2,\cdots,6,$$

$$\Upsilon_{1,j} = \bar{\sigma}_j \begin{bmatrix} \tilde{B}_f C_j + V_3 C_j & 0 & -\tilde{B}_f I_j - V_3 I_j \\ \tilde{B}_f C_j + V_5 C_j & 0 & -\tilde{B}_f I_j - V_5 I_j \\ V_7 C_j & 0 & -V_7 I_j \end{bmatrix},\ \Upsilon_{2,j} = \bar{\upsilon}_j \begin{bmatrix} V_3 C_j & 0 & -V_3 I_j \\ V_5 C_j & 0 & -V_5 I_j \\ V_7 C_j & 0 & -V_7 I_j \end{bmatrix},$$

$$\Upsilon_{3,j} = \rho_j \begin{bmatrix} \tilde{B}_f C_j & 0 & -\tilde{B}_f I_j \\ \tilde{B}_f C_j & 0 & -\tilde{B}_f I_j \\ 0 & 0 & 0 \end{bmatrix},\ \Upsilon_{4,j} = \bar{\sigma}_j \begin{bmatrix} \tilde{B}_f I_j + V_3 I_j \\ \tilde{B}_f I_j + V_5 I_j \\ V_7 I_j \end{bmatrix}$$

$$\Upsilon_{5,j} = \bar{\upsilon}_j \begin{bmatrix} V_3 I_j \\ V_5 I_j \\ V_7 I_j \end{bmatrix},\ \Upsilon_{6,j} = \rho_j \begin{bmatrix} \tilde{B}_f I_j \\ \tilde{B}_f I_j \\ 0 \end{bmatrix},\ j = 1,2,\cdots,r,$$

$$\overline{\Psi}_3 = diag\{\breve{N},\ \breve{N},\ \breve{N}\},\ N = \tilde{P} - V - V^T,\ \breve{N} = diag_r\{N\},$$

$$\overline{\Psi}_4 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ 0 & \Sigma_{22} & 0 & 0 \\ \Sigma_{31} & 0 & 0 & 0 \end{bmatrix},$$

$$\Sigma_{11} = \begin{bmatrix} V_1 A + \tilde{B}_f \bar{\Xi} C + V_3(\bar{\Xi} + \bar{\Delta})C & \tilde{A}_f & \tilde{B}_f(I - \bar{\Xi}) + V_3(I - \bar{\Xi} - \bar{\Delta}) \\ V_4 A + \tilde{B}_f \bar{\Xi} C + V_5(\bar{\Xi} + \bar{\Delta})C & \tilde{A}_f & \tilde{B}_f(I - \bar{\Xi}) + V_5(I - \bar{\Xi} - \bar{\Delta}) \\ V_6 A + V_7(\bar{\Xi} + \bar{\Delta})C & 0 & V_7(I - \bar{\Xi} - \bar{\Delta}) \end{bmatrix},$$

$$\Sigma_{12} = vec_d \left\{ \bar{\alpha}_i \begin{bmatrix} V_1 \\ V_4 \\ V_6 \end{bmatrix} \right\},\ \Sigma_{13} = \begin{bmatrix} V_1 D \\ V_4 D \\ V_6 D \end{bmatrix},$$

$$\Sigma_{14} = \begin{bmatrix} \tilde{B}_f \bar{\Xi} + V_3(\bar{\Xi} + \bar{\Delta}) \\ \tilde{B}_f \bar{\Xi} + V_5(\bar{\Xi} + \bar{\Delta}) \\ V_7(\bar{\Xi} + \bar{\Delta}) \end{bmatrix},\ \Sigma_{22} = diag_d \left\{ \vartheta_i \begin{bmatrix} V_1 \\ V_4 \\ V_6 \end{bmatrix} \right\},$$

$$\Sigma_{31} = \begin{bmatrix} L & -\tilde{L}_f & 0 \end{bmatrix},\ \overline{\Psi}_5 = diag\{N,\ \widehat{N}, -I\},\ \widehat{N} = diag_d\{N\}.$$

Then filter matrices in the form of (3.6) can be obtained as

$$A_f = V_2^{-1} \tilde{A}_f,\ B_f = V_2^{-1} \tilde{B}_f,\ L_f = \tilde{L}_f. \tag{3.33}$$

Proof From Lemma 3.3, it leads to the equivalence between (3.25) and the following inequality:

$$\begin{bmatrix} -\bar{P} & * & * & * & * & * & * & * & * & * \\ -\breve{U} & -\Lambda & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * \\ 0 & 0 & 0 & -\gamma^2 I & * & * & * & * & * & * \\ \breve{R}\breve{C}_1 & 0 & 0 & \breve{R}\breve{B}_1 & \breve{M} & * & * & * & * & * \\ \breve{R}\breve{C}_2 & 0 & 0 & \breve{R}\breve{B}_2 & 0 & \breve{M} & * & * & * & * \\ \breve{R}\breve{C}_{12} & 0 & 0 & \breve{R}\breve{B}_{12} & 0 & 0 & \breve{M} & * & * & * \\ R\bar{A} & R\bar{G} & R\bar{\Gamma}_w & R\bar{\Gamma}_v & 0 & 0 & 0 & M & * & * \\ 0 & \hat{R}G & 0 & 0 & 0 & 0 & 0 & 0 & \hat{M} & * \\ \bar{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{3.34}$$

where $M = P - R - R^T$, $\breve{M} = diag_r\{M\}$, $\hat{M} = diag_d\{M\}$, $\breve{R} = diag_r\{R\}$, $\hat{R} = diag_d\{R\}$.

Then, we introduce the partition [37]:

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & 0 & R_{33} \end{bmatrix}$$

and assume that R_{12} is nonsingular. Define

$$S = \begin{bmatrix} I & 0 & 0 \\ 0 & R_{12}R_{22}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}$$

and let

$$\tilde{P} = SPS^T, V = SRS^T = \begin{bmatrix} V_1 & V_2 & V_3 \\ V_4 & V_2 & V_5 \\ V_6 & 0 & V_7 \end{bmatrix}$$

Performing congruence transformation to (3.34) by $\tilde{J} = diag\{S^T, I, I, I, \underbrace{S^T, \cdots, S^T}_{3r+1+d}, I\}$, we can obtain (3.32) by defining $\tilde{A}_f = R_{12}A_f R_{22}^{-T} R_{12}^T$, $\tilde{B}_f = R_{12}B_f$, $\tilde{L}_f = L_f R_{22}^{-T} R_{12}^T$, and $V_1 = R_{11}$, $V_2 = R_{12}R_{22}^{-T}R_{12}^T$, $V_3 = R_{13}$, $V_4 = R_{12}R_{22}^{-1}R_{21}$, $V_5 = R_{12}R_{22}^{-1}R_{23}$, $V_6 = R_{31}, V_7 = R_{33}$.

Further, it is noted that the parameters of the filter satisfy the following equation:

$$\begin{bmatrix} A_f & B_f \\ L_f & 0 \end{bmatrix} = \begin{bmatrix} R_{12}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}_f & \tilde{B}_f \\ \tilde{L}_f & 0 \end{bmatrix} \begin{bmatrix} R_{22}^{-T}R_{12}^T & 0 \\ 0 & I \end{bmatrix}$$

The discrete-time transfer function from y_k to $\hat{z}_k$ is represented by

$$T_{\hat{z}y} = L_f(zI - A_f)^{-1}B_f = \tilde{L}_f(zI - V_2^{-1}\tilde{A}_f)^{-1}V_2^{-1}\tilde{B}_f$$

Therefore, the parameters of the filter are given by

$$A_f = V_2^{-1}\tilde{A}_f,\ B_f = V_2^{-1}\tilde{B}_f,\ L_f = \tilde{L}_f.$$

The proof is completed.

It is worth mentioning that the inequality (3.32) is not only linear with respect to matrix variables $\tilde{P}$, V, $\tilde{A}_f$, $\tilde{B}_f$, $\tilde{L}_f$ and $\lambda_1, \cdots, \lambda_d$, but also linear with respect to scalar γ^2. So, the minimum H_∞ attenuation level γ_{min} can be obtained by solving the convex optimization problem:

Problem 3.1

$$\min_{\lambda_1,\cdots,\lambda_d>0,\tilde{P}>0,V,\tilde{A}_f,\tilde{B}_f,\tilde{L}_f} \gamma^2$$

subject to (3.32)

The computation procedures of the designed filter given in Theorem 3.3 can be summarized as follows:

Step 1: Given the definitions of decision variables $\tilde{P}$, $V(V_1, V_2, \cdots, V_7)$, $\widetilde{A}_f$, $\tilde{B}_f$, $\tilde{L}_f$, $\lambda_1, \cdots, \lambda_d$, and determine their dimensions according to the considered network-based system.

Step 2: Write the LMI terms according to (3.32) by using MATLAB LMI toolbox.

Step 3: If we only want to obtain the feasible solutions of decision variables, then for a given disturbance rejection attenuation level γ, solve the LMI (32) by applying the "feasp" solver. Else, if we not only want to obtain the feasible solutions of decision variables, but also want to obtain the minimum disturbance rejection attenuation level γ_{min}, we can take γ as a decision variable and solve the Problem 3.1 by applying the "mincx" solver.

Step 4: Substituting the obtained variables $\tilde{A}_f$, $\tilde{B}_f$, $\tilde{L}_f$, and V_2 into (3.33), the desired filter parameters A_f, B_f, L_f can be got.

Corollary 3.1 We can also adopt the method of partitioning the matrix P directly to obtain the filter matrices. Along the similar line of the proof of Theorem 3.3, let $=$ $\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & 0 \\ P_{13}^T & 0 & P_{33} \end{bmatrix}$, and assume that P_{12} is nonsingular. Define $\bar{S} = \begin{bmatrix} I & 0 & 0 \\ 0 & P_{12}P_{22}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}$, performing congruence transformation to (3.25) by $\bar{J} = diag\{\bar{S}^T, I, I, I, \underbrace{\bar{S}^T, \cdots, \bar{S}^T}_{3r+1+d}, I\}$, one can have the corresponding results $A_f = \bar{V}_2^{-1}\hat{A}_f$, $B_f = \bar{V}_2^{-1}\hat{B}_f$, $L_f = \hat{L}_f$ by defining $\bar{V}_1 = P_{11}$, $\bar{V}_2 = P_{12}P_{22}^{-1}P_{12}^T$, $\bar{V}_3 = P_{13}^T$, $\bar{V}_4 = P_{33}$ and $\hat{A}_f = P_{12}A_fP_{22}^{-1}P_{12}^T$, $\hat{B}_f = P_{12}B_f$, $\hat{L}_f = L_fP_{22}^{-1}P_{12}^T$, which are omitted here for conciseness.

Remark 3.5 Theorem 3.3 gives a sufficient condition to obtain the H_∞ filter matrices by solving the feasibility of an LMI in (3.32) for the network-based system (3.1) with randomly occurred nonlinearities where different channels have different delay rates and packet loss rates. We introduce the slack variable R in Theorem 3.3 which is not required to be positive definite, so it can increase the flexibility in determining the matrices. Moreover, it is not difficult to see that the method we adopted includes the method of Corollary 3.1 as a special case. Therefore, it can reduce the design conservativeness.

Remark 3.6 It is reported that the computational complexity of the LMI-based algorithms depends polynomially on the total number of decision variables. That is, the number $N(\varepsilon)$ of flops needed to compute an ε-accurate solution is bounded by $O(XN^3\log(\Upsilon/\varepsilon))$, where X is the total row size of the LMI system, N is the total number of scalar decision variables, Y is a data-dependent scaling factor, and ε is relative accuracy set for algorithm [38]. For system (3.1), the variable dimensions can be seen from $x_k \in \mathbb{R}^n$, $\tilde{y}_k \in \mathbb{R}^r$, $z_k \in \mathbb{R}^m$, $w_k \in \mathbb{R}^q$ and $v_k \in \mathbb{R}^r$. From Theorem 3.3, we have $X = 6nr + 3nd + 3r^2 + 4n + rd + 3r + q + m$, $N = 6n^2 + \frac{3}{2}r^2 + 6nr + mn + \frac{r}{2} + n + d$, therefore, the time complexity of our algorithm can be represented as $O(n^7r + n^7d + n^6rd + r^8 + r^7n + r^7d + r^6nd)$. Take a closer look at the Theorem 3.3 and Corollary 3.1, it is easy to see that the values of X are equal, but the value of N in Theorem 3.3 is larger than Corollary 3.1, so the computing time of Theorem 3.3 is longer than Corollary 3.1 although the precise is better. In a word, the fewer conservative results are obtained at the expense of increased computational cost.

3.5 Simulation example

In this section, we will present two examples to illustrate the effectiveness and applicability of our proposed filtering design method.

Example 1 Consider a network-based system in the form of (3.1) with the following parameters [39]:

$$A = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.1 & -0.3 & 0.1 \\ 0.1 & 0 & -0.2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0.8 & 0.7 \\ -0.6 & 0.9 & 0.6 \\ 0.2 & 0.1 & 0.1 \end{bmatrix},$$

$$D = \begin{bmatrix} -0.2 & 0 & 0.1 \\ -0.1 & 0.1 & 0.1 \\ 0 & 0.2 & 0.1 \end{bmatrix}, L = [-0.1 \quad 0 \quad 0.1],$$

The randomly occurred nonlinear functions are chosen as [20,40]:

$$f_1(x_k) = \begin{bmatrix} \tanh(-x_{1k}) + 0.2x_{1k} + 0.1x_{2k} + 0.3x_{3k} \\ 0.1x_{1k} - \tanh(x_{2k}) + 0.2x_{2k} \\ 0.1x_{1k} + 0.2x_{3k} - \tanh(x_{3k}) \end{bmatrix},$$

$$f_2(x_k) = \begin{bmatrix} \tanh(-x_{1k}) + 0.3x_{1k} + 0.2x_{2k} + 0.1x_{3k} \\ 0.1x_{1k} - \tanh(x_{2k}) + 0.2x_{2k} + 0.3x_{3k} \\ 0.1x_{1k} + 0.2x_{3k} - \tanh(x_{3k}) \end{bmatrix},$$

It can be verified that

$$U_{11}=\begin{bmatrix}-0.8 & 0.1 & 0.1\\ 0.1 & -0.8 & 0\\ 0.1 & 0 & -0.8\end{bmatrix},\ U_{12}=\begin{bmatrix}0.2 & 0.1 & 0.1\\ 0.1 & 0.2 & 0\\ 0.1 & 0 & 0.2\end{bmatrix},$$
$$U_{21}=\begin{bmatrix}-0.7 & 0.2 & 0.1\\ 0.1 & -0.8 & 0.3\\ 0.1 & 0 & -0.8\end{bmatrix},\ U_{22}=\begin{bmatrix}0.3 & 0.2 & 0.1\\ 0.1 & 0.2 & 0.3\\ 0.1 & 0 & 0.2\end{bmatrix},$$

Note that the above system is with three outputs and the packet loss probability and the possible one-step transmission delay probability for these three channels may be different. To demonstrate the effectiveness of the proposed H_∞ filtering design approach, we first compare the Theorem 3.3 with Corollary 3.1 under the different nonlinearity scenarios when the probabilities of random variables denoting the packet loss rate and delay rate are fixed by $\bar{\xi}_1=0.8$, $\bar{\xi}_2=0.6$, $\bar{\xi}_3=0.9$, $\bar{\beta}_1=0.5, \bar{\beta}_2=0.7$, and $\bar{\beta}_3=0.3$. The results are summarized in Table 3.1.

Table 3.1 Comparison of minimum H_∞ performance under different cases.

Method	Cases	$\bar{\alpha}_1=0.1$, $\bar{\alpha}_2=0.1$	$\bar{\alpha}_1=0.2$ $\bar{\alpha}_2=0.2$	$\bar{\alpha}_1=0.3$ $\bar{\alpha}_2=0.3$
Theorem 3.3	γ_{min}	0.0436	0.0460	0.0518
Corollary 3.1	γ_{min}	0.0438	0.0461	0.0520
Method	Cases	$\bar{\alpha}_1=0.4$ $\bar{\alpha}_2=0.4$	$\bar{\alpha}_1=0.5$ $\bar{\alpha}_2=0.5$	$\bar{\alpha}_1=0.6$ $\bar{\alpha}_2=0.6$
Theorem 3.3	γ_{min}	0.0618	0.0769	0.0992
Corollary 3.1	γ_{min}	0.0625	0.0786	0.1055
Method	Cases	$\bar{\alpha}_1=0.7$ $\bar{\alpha}_2=0.7$	$\bar{\alpha}_1=0.8$ $\bar{\alpha}_2=0.8$	$\bar{\alpha}_1=0.9$ $\bar{\alpha}_2=0.9$
Theorem 3.3	γ_{min}	0.1313	0.1737	0.2215
Corollary 3.1	γ_{min}	0.1655	Infeasible	Infeasible

From Table 3.1, we can see that when the probability of nonlinearity occurred is becoming larger, the minimum disturbance attenuation level γ_{min} is becoming larger, *i.e.*, the H_∞ performance is of degradation. However, the results by using Theorem 3.3 are smaller than that of the Corollary 3.1. That is to say, the introduction of the slack variable has much better performance over the common matrix partition method. When the probability of nonlinearity occurred is nearly to 1, the LMI in Corollary 3.1 is infeasible. But the results obtained by Theorem 3.3 are still good. It verifies that the more serious the randomly nonlinearities are, the more superior our algorithm is.

Moreover, in order to give the effect of the rates of packet dropout and transmission delay on the H_∞ performance, we consider the cases of $\bar{\xi}_1=\bar{\xi}_2=\bar{\xi}_3=$

0.6, $\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}_3 = 0.7$ and $\bar{\xi}_1 = \bar{\xi}_2 = \bar{\xi}_3 = 0.9$, $\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}_3 = 0.3$ when the nonlinearities occurred probabilities are fixed by $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_2 = 0.7$. By solving the problem of Problem 3.1, we can get the optimal H_∞ attenuation level are $\gamma_{min} = 0.1315$ and $\gamma_{min} = 0.1300$, respectively. It is to say that when the on-time rate for a packet received by the filter is larger (or in other words, the packet dropout rate and the transmission delay rate are smaller), the optimal disturbance attenuation level is smaller.

We further give the simulation results when the initial values are $x_0 = [0.1 \quad -0.2 \quad -0.1]^T$ and $\hat{x}_0 = [0 \quad 0 \quad 0]^T$. The other parameters are also given as $\bar{\xi}_1 = 0.8$, $\bar{\xi}_2 = 0.6$, $\bar{\xi}_3 = 0.9$, $\bar{\beta}_1 = 0.5$, $\bar{\beta}_2 = 0.7$, $\bar{\beta}_3 = 0.3$ and $\bar{\alpha}_1 = 0.7$, $\bar{\alpha}_2 = 0.7$. The external disturbance w_k and v_k are assumed to be mutually independent white noise with zero-mean and variance I_3. By using MATLAB toolbox to solve the feasibility of LMI in Theorem 3.3, we can obtain the desired H_∞ filter parameters as follows

$$A_f = \begin{bmatrix} 0.0920 & -0.0886 & 0.0466 \\ -0.0079 & 0.1012 & -0.1364 \\ -0.0265 & -0.0857 & 0.1104 \end{bmatrix}, B_f = \begin{bmatrix} -0.0175 & -0.0559 & -0.0100 \\ -0.0091 & 0.0761 & -0.0279 \\ -0.0265 & -0.0857 & 0.1104 \end{bmatrix},$$

$$L_f = [0.1724 \quad -0.0132 \quad -0.1842].$$

The measurements of each channel received by the estimator can be obtained by the values of random variables of $\xi_{i,k}$ and $\beta_{i,k} (i = 1,2,3)$ according to Equation (3.5), which is shown in Figure 3.2. Let $\tilde{y}_{i,k-1}$ represent the output of the ith sensor arriving at the estimator through the networks with one-step delay, $\tilde{y}_{i,k}$ represent the output arriving on time, and 0 denote the packet lost during the transmission but in such a case we use the latest received measurement to design the filter. For example, at the time of $i = 40$, the measurement output of the first channel is received by the estimator with one-step delay, and it is the same case with the second channel, but for the third channel, the measurement output is received by the estimator on time. The estimate result of z_k is shown in Figure 3.3, from which we can see that the designed filter can estimate the state well even though the system contains some uncertainties including time-varying delays, consecutive packet losses and randomly occurred nonlinearities.

In addition, let the disturbance be $w_k = 2e^{-0.1k}\sin(0.1\pi k)$, and v_k is assumed to be white noise with zero-mean and variable $0.1 I_3$. The other parameters are unchanged. By solving Problem 3.1, we can get the optimal H_∞ attenuation level is $\gamma_{min} = 0.1313$ and the filter parameters are as follows

$$A_f = \begin{bmatrix} 0.1578 & 0.1000 & 0.1821 \\ -0.2811 & 0.0772 & -0.2805 \\ 0.1602 & -0.1775 & 0.1294 \end{bmatrix}, \ B_f = \begin{bmatrix} -0.3000 & 0.3542 & -0.1736 \\ 0.1322 & -0.0122 & 0.0673 \\ 0.1018 & -0.4088 & 0.0219 \end{bmatrix}$$

$$L_f = [0.1008 \quad 0.0011 \quad -0.0998]$$

The estimation error is given in Figure 3.4. It can be seen that the error dynamics can convergence to zero fast. It is to say the filtering error system is stable, which also shows the effectiveness of our approach.

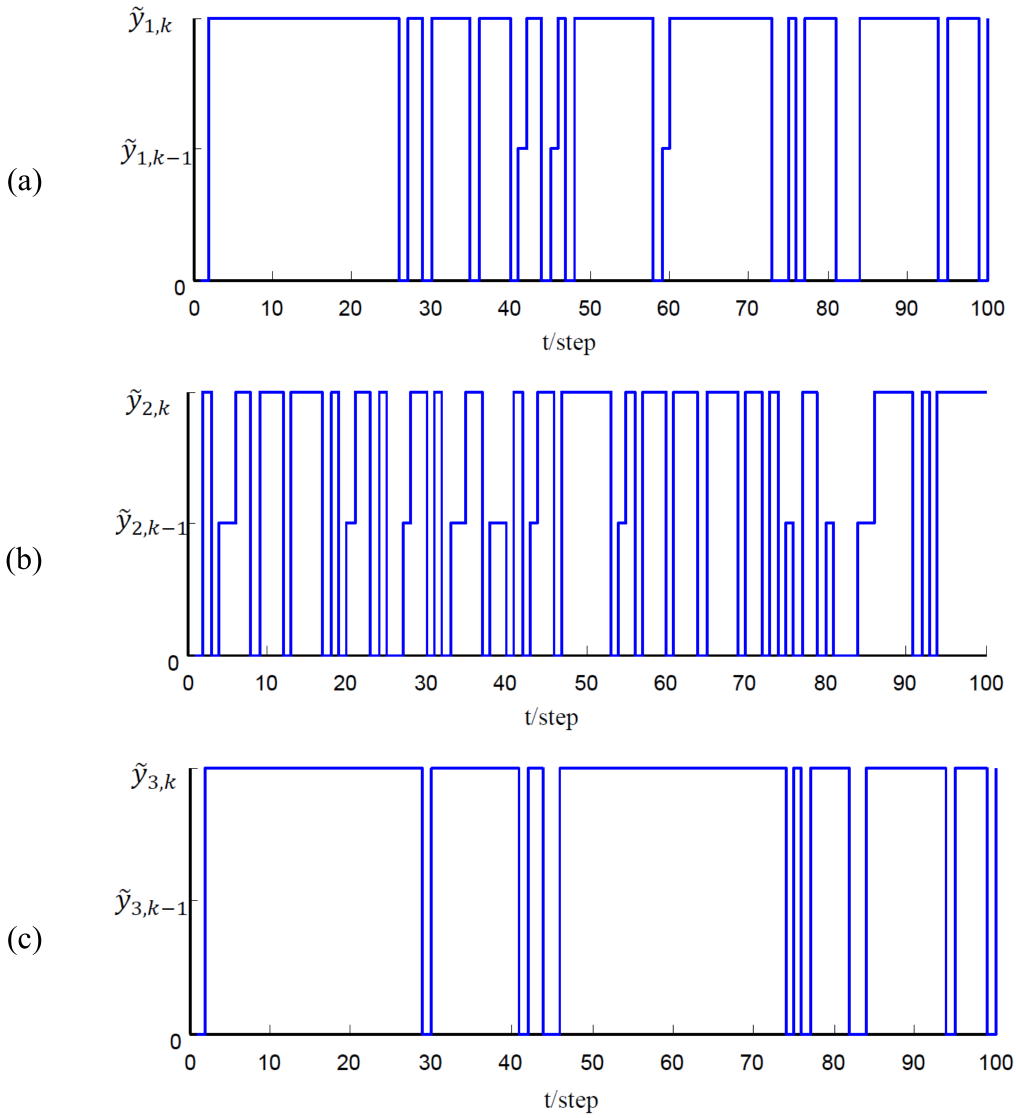

Figure 3.2 The measurements received by estimator. (a) Measurements received by estimator for the first channel; (b) Measurements received by estimator for the second channel; (c) Measurements received by estimator for the third channel.

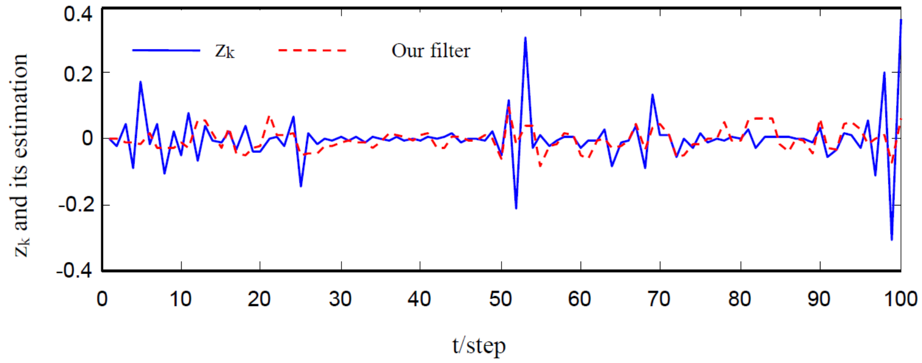

Figure 3.3 Signal z_k and its estimate $\hat{z}_k$.

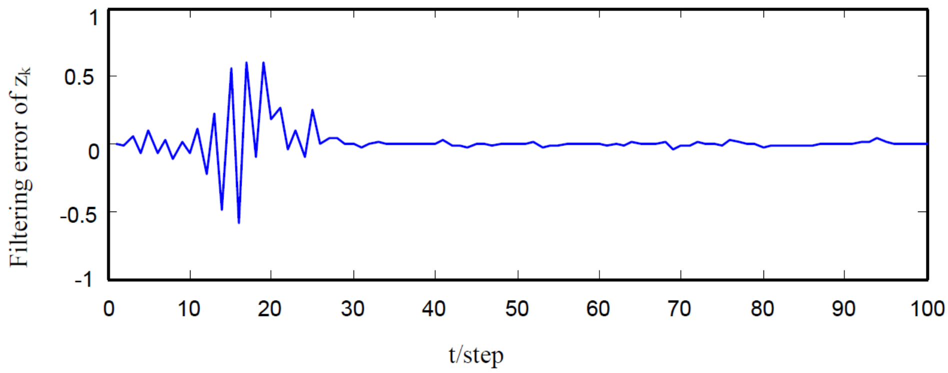

Figure 3.4 The filtering error response.

Example 2 In this example, we present the F-404 aircraft engine system to demonstrate the applicability of the proposed filtering approach. Setting the sampling time as $T = 0.05\ s$, we obtain the following discretized system matrix A and the measurement output matrix C as [40]:

$$A = \begin{bmatrix} 0.9270 & 0 & 0.1214 \\ 0.0082 & 0.9800 & -0.0189 \\ 0.0155 & 0 & 0.8885 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and the matrices D and L are set as

$$D = [0.15 \quad 0.05 \quad 0.12]^T,\ L = [0.1 \quad 0.1 \quad -0.1].$$

The probabilities of the random variables are given by $\bar{\xi}_1 = 0.7$, $\bar{\xi}_2 = 0.6$, $\bar{\beta}_1 = 0.5$, $\bar{\beta}_2 = 0.9$ and $\bar{\alpha}_1 = 0.8$, $\bar{\alpha}_2 = 0.8$. The other parameters are set as the same in Example 1. With the above parameters and by using the MATLAB LMI toolbox, let $\gamma = 0.9$, then we obtain

$$A_f = \begin{bmatrix} 0.1784 & -0.1952 & -0.0389 \\ -0.1444 & 0.2886 & -0.0561 \\ 0.4081 & 0.3471 & 0.5761 \end{bmatrix},\ B_f = \begin{bmatrix} -1.4644 & 0.4774 \\ 0.5433 & -1.1258 \\ 1.0504 & 0.1133 \end{bmatrix},$$

$$L_f = [-0.1118 \quad -0.1133 \quad 0.1150].$$

The filtering error is shown in Figure 3.5, from which we can see that the filtering error convergence to zero in the absence of disturbances.

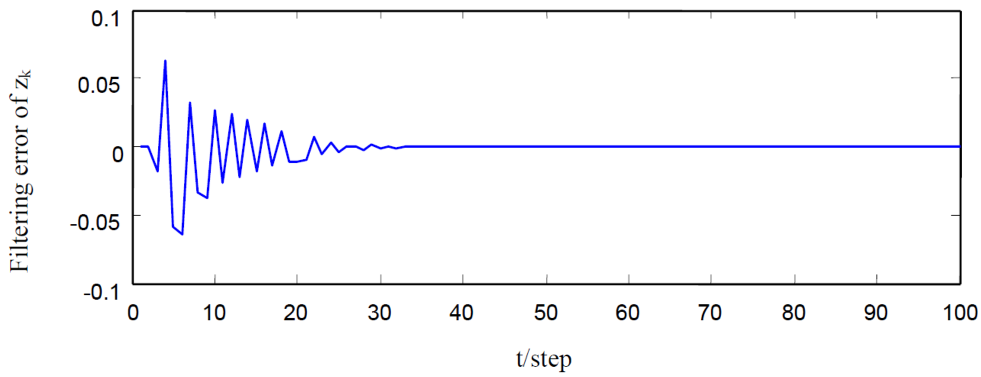

Figure 3.5 The filtering error response.

3.6 Conclusions

In this chapter, the H_∞ filtering problem has been investigated for a class of multiple channel networked systems with sector-bounded nonlinearities which occur in a random way. Due to the unreliable network medium, the time-varying delays and consecutive packet losses are involved where different channels have different time delay rates and packet loss rates. Two diagonal matrices whose leading diagonal elements are Bernoulli distributed stochastic variables have been introduced to model the random phenomena of packet losses and transmission delays of multiple channels. Then, a sufficient condition is derived such that the filtering error system is exponentially stable in the mean square and the H_∞ performance constraint is achieved for all nonzero exogenous disturbances under the zero-initial condition. The illustrative examples have been used to demonstrate the effectiveness and applicability of the filtering design approach we presented. The future research topic is to investigate the fuzzy filter design for the multiple channel systems with individual packet dropout rates and time delay rates because the T-S model has the strong ability to approximate the nonlinear functions. Additionally, the other research topic is to design the robust filter for systems with randomly occurred uncertain parameters.

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Chapter 4 H_∞ Filtering for Network-based Systems with Delayed Measurements, Packet Losses and Randomly Varying Nonlinearities

4.1 Introduction

The state estimation has been a focus of research for several decades because of their important engineering applications in manufacturing, process control, mechatronics, biomedical systems, and so on. Among a variety of existing state estimation methods, the H_∞ filtering approach has gained particular research attention due to its capability of minimizing the largest energy gain of the estimation error for all initial conditions and noises. A large number of results have been reported in the literatures, see e.g., [1-5] and the references therein.

In recent years, one of the most significant trends is the increasing application of networks in modern dynamical processes such as advanced aircraft, automotive and manufacturing processes. The introduction of networks brings a lot of advantages, e.g., low cost, high reliability, reduced weight and power requirements, simple installation and maintenance, *etc*. However, the bandwidth of networks is limited and the external environment is complex, such that the NCS-related challenging problems arise inevitably which include, but are not limited to, packet losses, communication delays and varying nonlinearities. At present, more and more researchers have paid their attention to how to conduct these random phenomena in a unified framework [6-8]. The H_∞ filtering problem is investigated for the nonlinear stochastic systems subject to sensor saturation over unreliable communication channel in [6] where the effect of output logarithmic quantization and data loss is considered together. In [7], the robust and reliable H_∞ filter is designed for a class of T-S model-based nonlinear systems where the signal transfer of the closed-loop system is under a networked communication scheme and therefore is subject to time delay, packet loss, and/or packet out of order. In [8], the H_∞ filtering problem is addressed for a class of discrete-time stochastic systems subject to randomly occurring gain variation, channel fading, as well as randomly occurring nonlinearity.

It is well known that, among these random phenomena, one of the important issues is the network-induced delay which is usually caused by limited bit rates of the communication channel. Up to now, more and more efforts have been focused on the filtering problems for various time-delay systems [9-12] without packet losses. In a real network environment, if the packet is with a delay longer than a certain pre-determined number, one possible strategy is to discard the packet and treat it as a packet dropout, which could deteriorate the filter performance or even lead to the instability of the filter dynamics. Furthermore, the transmission delays may lead to the out of sequence of measurements received by the remote data processing center/filter. In most practical cases of NCSs, the time-stamp technique [13-16] is often used to get the size of the transmission delay or whether a packet is lost at a certain time. Moreover, the data packets arrived at receiver can be reordered by using the time-stamp technique. For a finite maximum delay, we use a finite memory buffer in the receiver side and there will be multiple data packets arrived at the buffer at each time instant. However, in the above literatures, only the latest received data stored up in the buffer is used to design the filter or controller which can waste some aforementioned information.

On the other hand, nonlinearities are recognized to exist universally in practical systems [17]. It is well known that many practical systems are influenced by additive nonlinear disturbances from environments. If not properly handled, the nonlinearities will cause great degradation of the system performance or even lead to instability. The nonlinear disturbances themselves may experience random abrupt changes due to the variations and failures arising from network-induced phenomena. In other words, the type and intensity of the nonlinearities could be changeable in a probabilistic way, which gives rise to the so-called randomly varying nonlinearities. In nowadays prevalent research for the NCSs, much attention has been focused on the nonlinearity caused by the time invariant state [8,18,19]. Unfortunately, this is not always the case in practice. If the nonlinearity is caused by the time-delayed state, it cannot be handled by this model.

Motivated by the above discussion, this chapter focuses on the H_∞ filtering problem for networked stochastic systems subject to time varying nonlinearities, delayed measurements and packet losses. The main contributions of this chapter are as follows: (1) the system model considered is comprehensive as it takes into account randomly delayed measurement, packet losses, time varying nonlinearities with random occurrence, which can reflect the reality more closely; (2) a finite memory buffer is used at the receiver side and a full-order nonlinear H_∞ filter is derived by using all the time-stamped data packets in the buffer, which can avoid the aforementioned information waste effectively; (3) by constructing a new delay-dependent combined with nonlinearity-dependent Lyapunov-Krasovskii functional, a nonlinear filter is designed such that, for all possible delayed measurements, packet dropouts and randomly varying nonlinearities, the dynamic of the estimation error is guaranteed to be asymptotically stable and the filtering error satisfies H_∞ performance constraint for all nonzero exogenous disturbances under zero-initial condition. At the end of this chapter, an example is given to show the effectiveness of the proposed design approach.

4.2 Problem formulation

In this chapter, we consider the H_∞ filtering problem for a class of discrete-time stochastic nonlinear systems which can be represented by the following equations:

$$\begin{cases} x_{k+1} = Ax_k + \vartheta_k f(x_{k-\tau_k}) + (1-\vartheta_k) g(x_{k-h_k}) + \Gamma_1 w_k \\ y_k = Cx_k + \Gamma_2 w_k \\ z_k = Lx_k \end{cases} \tag{4.1}$$

where $x_k \in \mathbb{R}^n$ is the system state; $y_k \in \mathbb{R}^r$ is the measured output; $z_k \in \mathbb{R}^m$ is the signal to be estimated; $w_k \in \mathbb{R}^q$ denotes the exogenous disturbance signal belonging to $l_2[0,\infty)$, and A, C, Γ_1, Γ_2 and L are known, real, constant matrices with suitable dimensions. The positive integers τ_k and h_k denote the varying time delays satisfying $0 \le \underline{\tau} \le \tau_k \le \bar{\tau}$, $0 \le \underline{h} \le h_k \le \bar{h}$,where $\underline{\tau}$, $\underline{h}$ and $\bar{\tau}$, $\bar{h}$ are known positive integers representing the minimal and maximal delays, respectively. The Bernoulli distributed white sequence ϑ_k is introduced to account for the random nature of the occurrence of the nonlinearity for system Equation (4.1). It is assumed that ϑ_k obeys the probability distributions as $\text{Prob}\{\vartheta_k = 1\} = \bar{\vartheta}$, $\text{Prob}\{\vartheta_k = 0\} = 1 - \bar{\vartheta}$, where $\bar{\vartheta} \in [0,1]$ is a known constant. It means that

$$\vartheta_k = \begin{cases} 1, & system\ (4.1) experiences\ nonlinear\ function\ f(\cdot) \\ 0, & system\ (4.1) experiences\ nonlinear\ function\ g(\cdot) \end{cases}$$

Here the nonlinear functions $f(\cdot)$ and $g(\cdot)$ are assumed to satisfy the following sector-bounded conditions:

Assumption 4.1 $f(0) = 0$, $g(0) = 0$.

Assumption 4.2 For $\forall \mathrm{x}, \mathrm{y} \in \mathbb{R}^n$, there exist matrices U_1, U_2, U_3 and U_4 such that

$$[f(x) - f(y) - U_1(x - y)]^T[f(x) - f(y) - U_2(x - y)] \leq 0 \tag{4.2}$$

$$[g(x) - g(y) - U_3(x - y)]^T[g(x) - g(y) - U_4(x - y)] \leq 0 \tag{4.3}$$

Remark 4.1 The nonlinearities with time varying delays are considered in [20] but only one type of nonlinearities occurs randomly. In this chapter, the nonlinearities $f(x_{k-\tau_k})$ and $g(x_{k-h_k})$ in model (4.1) are time varying and randomly switchable by means of their types or intensities based on their individual probability distributions. In [18,19], the multiple sector-like nonlinearities $\sum_{i=1}^{d} \alpha_{i,k} f_i(x_k)$ are considered. But if all the Bernoulli distributed random variables $\alpha_{i,k} = 0$, there is no nonlinearities occur at time k. However, it is well known that nonlinearities ubiquitously exist in practical systems. So, in model (4.1) presented in this chapter, when $f = g$, although the stochastic property disappears, the nonlinearity also exists. In [21], the randomly nonlinearities are taken as $\delta_k f(x_k) + (1 - \delta_k) g(x_k)$ which are caused by the time invariant state. It is only a special case in the model (4.1) we presented where the time-varying delays τ_k and h_k are equal to zero.

For the network-based filtering problem, the transmission of a packet from the sensor to the remote filter can be divided into two stages: sensing and networking. The output data of the plant are sampled first by the sensor. Up to now, the sensor has two triggering modes, that is, the event-triggered [22,23] and the clock-triggered. Here, we assume that the sensor is clock-triggered and the sampling process is at a synchronous rate. Then the sampled data are packed and transmitted through the network to the remote filter. To avoid the network congestion, each packet at sensor side is assumed to be sent only once. Due to the unreliable network media, the random phenomena will occur during the transmission such as the missing measurements and delays which lead to the packets received by the filter not in a normal chronological order. By applying the time-stamp technique, we can know the maximum delay and the out-of-sequence packets also can be reordered at the receiver side. Due to the transmission delay, there will be multiple data packets arrived at the buffer at each time instant which are all used in the following filter design to avoid the aforementioned information waste effectively.

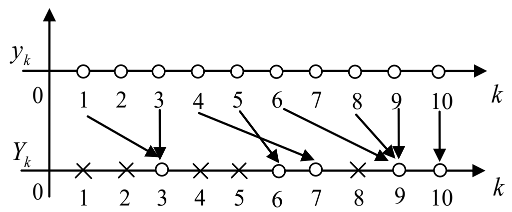

Figure 4.1 Data transmission case in the receiver with a finite memory buffer.

We depict the transmission process in Figure 4.1 where the maximum delay is assumed to be 3. It is worth mentioning that we denote the packets arrived at the receiver side between time $k-1$ and k as received at the time instant k. From Figure 4.1, we can see that y_3, y_9 and y_{10} are received on time, y_5 and y_8 are delayed by one step, y_1 is delayed by two steps, y_4 and y_6 are delayed by three steps, y_2 and y_7 are lost. Furthermore, it can be easily seen that the data processing center/filter may receive one or multiple packets or nothing at each time. For example, there are three packets received by the filter at $k=9$, and there are two packets received by the filter at $k=3$, *etc*. These phenomena can be governed by a sequence of stochastic variables. Thus, the following model for measurements received by the filter is constructed [24]:

$$Y_k = \begin{bmatrix} \xi_k^{(0)} y_k \\ (1-\xi_{k-1}^{(0)})\xi_k^{(1)} y_{k-1} \\ \vdots \\ \prod_{i=0}^{d-1}(1-\xi_{k-d+i}^{(i)})\xi_k^{(d)} y_{k-d} \end{bmatrix} \tag{4.4}$$

where $Y_k \in \mathbb{R}^{(d+1)r}$ is the measurement received by the filter. $\xi_k^{(i)} (0 \le i \le d)$ are a group of Bernoulli distributed white sequences which are uncorrelated with each other and also uncorrelated with ϑ_k. The probabilities of $\xi_k^{(i)} (0 \le i \le d)$ satisfy that $\text{Prob}\{\xi_k^{(i)} = 1\} = \bar{\xi}_i$ and $\text{Prob}\{\xi_k^{(i)} = 0\} = 1 - \bar{\xi}_i$. d denotes the maximal transmission delay.

Remark 4.2 The latest references [25,26] modeled the NCS by applying a group of Bernoulli distributed random variables to describe the possible transmission delays and packet losses. However, only the latest received packet is used to design the filter in [25]. Although the multiple data packets at the receiver side are used in [26], the time delay considered is only one-step. The model (4.4) is first presented in [24] including the special case of only delays but no packet losses when $\xi_k^{(d)} = 1$ [10]. However, only the white noise can be conducted. When the noise statistics is not exactly known and the disturbance is not random, the method in [24] cannot be used.

To conduct conveniently for the subsequent discussion, some new variables are introduced. Let

$$\beta_k^{(0)} = \xi_k^{(0)},\ \beta_k^{(j)} = \prod_{i=0}^{j-1}(1-\xi_{k-j+i}^{(i)})\,\xi_k^{(j)}\ \ (1 \le j \le d),$$

then, we have

$$E\{\beta_k^{(0)}\} = \bar{\xi}_0 = \bar{\beta}_0,\ E\{\beta_k^{(j)}\} = \prod_{i=0}^{j-1}(1-\bar{\xi}_i)\,\bar{\xi}_j = \bar{\beta}_j,$$

$$E\{(\beta_k^{(j)} - \bar{\beta}_j)^2\} = \bar{\beta}_j(1-\bar{\beta}_j) = \sigma_j^2,$$

$$E\{(\beta_k^{(j)}\beta_k^{(l)}\} = \bar{\beta}_j\bar{\beta}_l\ (j \neq l),\ \beta_k^{(j)}\beta_{k+l-j}^{(l)} = 0 (j \neq l)$$

The purpose of this chapter is to design a filter for discrete- time stochastic system (4.1) with delayed measurements, packet dropouts and random varying nonlinearities based on the measurements Y_k received by the filter. To accomplish this, we construct the nonlinear filter under the consideration of the structure of (4.4) as follows:

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + \bar{\vartheta} f(\hat{x}_{k-\tau_k}) + (1-\bar{\vartheta}) g(\hat{x}_{k-h_k}) + \sum_{i=0}^{d} K^{(i)}\beta_k^{(i)} y_{k-i} - \sum_{i=0}^{d} K^{(i)}\bar{\beta}_i C\hat{x}_{k-i} \\ \hat{z}_k = L\hat{x}_k \end{cases} \tag{4.5}$$

where $\hat{x}_k \in \mathbb{R}^n$ and $\hat{z}_k \in \mathbb{R}^m$ are the estimates of x_k and z_k. $K^{(i)}$, $i = 0,1,\cdots d$ are the filter parameters to be determined.

Remark 4.3 It is noted that the designed filter of form (4.5) contains the random variables $\beta_k^{(i)}$ which are the indicators of y_{k-i} received or not. From model (4.4), it can be easily seen that the on-time rate for a packet received by the filter is $Prob\{\beta_k^{(0)} = 1\} = \bar{\beta}_0$, the one-step delay rate is $Prob\{\beta_k^{(1)} = 1\} = \bar{\beta}_1$, $\cdots$, and the d-step delay rate is $Prob\{\beta_k^{(d)} = 1\} = \bar{\beta}_d$, and the packet loss rate is $1 - \sum_{i=1}^{d}\bar{\beta}_i$.

Let $e_k = x_k - \hat{x}_k$, $\tilde{z}_k = z_k - \hat{z}_k$, $\tilde{f}_k = f(x_k) - f(\hat{x}_k)$, $\tilde{g}_k = g(x_k) - g(\hat{x}_k)$. By combination of system (4.1) and filter (4.5), the filtering error dynamic can be obtained as follows:

$$\begin{cases} e_{k+1} = Ae_k + (\vartheta_k - \bar{\vartheta}) f(x_{k-\tau_k}) + \bar{\vartheta}\tilde{f}_{k-\tau_k} - (\vartheta_k - \bar{\vartheta}) g(x_{k-h_k}) + (1-\bar{\vartheta})\tilde{g}_{k-h_k} + \Gamma_1 w_k \\ -\sum_{i=0}^{d} K^{(i)}\bar{\beta}_i C e_{k-i} - \sum_{i=0}^{d} K^{(i)}\bar{\beta}_i \Gamma_2 w_{k-i} - \sum_{i=0}^{d} K^{(i)}(\beta_k^{(i)} - \bar{\beta}_i)(Cx_{k-i} + \Gamma_2 w_{k-i}) \\ \tilde{z}_k = Le_k \end{cases} \tag{4.6}$$

Then, by setting $\eta_k = [x_k^T \quad e_k^T]^T$, we have

$$\begin{cases} \eta_{k+1} = \bar{A}\eta_k + \sum_{i=0}^{d}(\bar{A}_i + (\beta_k^{(i)} - \bar{\beta}_i)\tilde{A}_i)\eta_{k-i} + (\bar{F} + (\vartheta_k - \bar{\vartheta})\tilde{F})\breve{f}_{k-\tau_k} \\ +(\bar{G} + (\vartheta_k - \bar{\vartheta})\tilde{G})\breve{g}_{k-\tau_k} + \bar{\Gamma} w_k + \sum_{i=0}^{d}(\bar{\Gamma}_i + (\beta_k^{(i)} - \bar{\beta}_i)\tilde{\Gamma}_i) w_{k-i} \\ \tilde{z}_k = \bar{L}\eta_k \end{cases} \tag{4.7}$$

where

$$\breve{f}_k = [f^T(x_k) \quad \tilde{f}_k^T]^T,\ \breve{g}_k = [g^T(x_k) \quad \tilde{g}_k^T]^T,$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},\ \bar{F} = \begin{bmatrix} \bar{\vartheta} I & 0 \\ 0 & \bar{\vartheta} I \end{bmatrix},\ \bar{G} = \begin{bmatrix} (1-\bar{\vartheta})I & 0 \\ 0 & (1-\bar{\vartheta})I \end{bmatrix},$$

$$\bar{A}_i = \begin{bmatrix} 0 & 0 \\ 0 & -\bar{\beta}_i K^{(i)} C \end{bmatrix},\ \tilde{A}_i = \begin{bmatrix} 0 & 0 \\ -K^{(i)} C & 0 \end{bmatrix},\ i = 0,1,\cdots,d,$$

$$\tilde{F} = \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix},\ \tilde{G} = \begin{bmatrix} -I & 0 \\ -I & 0 \end{bmatrix},\ \bar{\Gamma} = \begin{bmatrix} \Gamma_1 \\ \Gamma_1 \end{bmatrix},\ \bar{L} = [0 \quad L],$$

$$\bar{\Gamma}_i = \begin{bmatrix} 0 \\ -\bar{\beta}_i K^{(i)} \Gamma_2 \end{bmatrix},\ \tilde{\Gamma}_i = \begin{bmatrix} 0 \\ -K^{(i)} \Gamma_2 \end{bmatrix},\ i = 0,1,\cdots,d.$$

Definition 4.1 For a given scalar $\gamma > 0$, the filtering error system (4.7) is said to be asymptotically stable with an H_∞ performance level γ if the following conditions hold:

(i) (asymptotical stability) The filtering error system (4.7) with $w_k = 0$ is said to be asymptotically stable in mean square if for any initial condition, the following holds:

$$\lim_{k\to\infty} E\{\|\eta_k\|^2\} = 0 \tag{4.8}$$

(ii) (H_∞ performance) Under zero initial condition and for any non-zero $w_k \in l_2[0,\infty)$, the filtering error $\tilde{z}_k$ satisfies

$$\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\sum_{i=0}^{d} \|w_{k-i}\|^2\}. \tag{4.9}$$

The problem under study is to design a full-order filter in the form of (4.5) such that in the presence of the packet losses, delayed measurements and the randomly varying nonlinearities for system (4.1), the asymptotical stability and H_∞ performance are both satisfied for the given disturbance attenuation level $\gamma > 0$.

4.3 Filtering performance analysis

In this section, we first prove the asymptotical stability of the filtering error system (4.7) with $w_k = 0$ and then analyze the H_∞ performance of the filtering process. The main results are given in Theorem 4.1 and Theorem 4.2, respectively.

Remark 4.4 It is worth mentioning that the state $x_{k-\tau_k}$ with time-delay $0 \le \underline{\tau} \le \tau_k \le \bar{\tau}$ and x_{k-h_k} with time-delay $0 \le \underline{h} \le h_k \le \bar{h}$ describes the real situation of network-transmission-induced delays in the field of networked control systems which includes the time-delay are equal to zeros as a special case. In addition, the nonlinearities with random occurrence we considered involved by the $x_{k-\tau_k}$ and x_{k-h_k} are more general. The coexisting time-varying delays and nonlinearities have resulted in the complexity of considered systems. In the following derivation, a delay-dependent and nonlinearity-dependent Lyapunov functional approach is developed. It is known that since the delay-dependent method includes information on the size of the delay, especially when the size of the delay is small, then the analysis and synthesis based on the delay-dependent approach are generally less conservative than the delay-independent ones. It is illustrated that the delay-dependent combined with nonlinearity-dependent Lyapunov functional proposed in this chapter can reduced the conservation.

Lemma 4.1 [27] For matrices A, $P_0 > 0$ and $P_1 > 0$, the following inequality:

$$A^T P_1 A - P_0 < 0$$

is equivalent to that there exists a matrix W such that

$$\begin{bmatrix} -P_0 & * \\ WA & P_1 - W - W^T \end{bmatrix} < 0$$

Theorem 4.1 Let the filter parameters $K^{(i)}$, $i = 0,1,\cdots d$ be given. The filtering error system (4.7) with $w_k = 0$ is asymptotically stable in the mean square, if there exist matrices $P > 0$, $S > 0$, $R > 0$, $Q_j > 0$ $(j = 1,2,\cdots d)$ and scalars $\lambda_1 > 0$, $\lambda_2 > 0$ such that

$$\Pi = \begin{bmatrix} \Pi_{11} & * & * & * & * \\ \Pi_{21} & (\bar{\tau} - \underline{\tau} + 1)S - \lambda_1 I & * & * & * \\ \Pi_{31} & 0 & (\bar{h} - \underline{h} + 1)R - \lambda_2 I & * & * \\ \Pi_{41} & 0 & 0 & \Pi_{44} & * \\ \Pi_{51} & 0 & 0 & \Pi_{54} & \Pi_{55} \end{bmatrix} < 0 \tag{4.10}$$

where

$$\Psi = \bar{\bar{A}}^T P \bar{\bar{A}} + \sigma_0^2 \tilde{A}_0^T P \tilde{A}_0 - P + \sum_{j=1}^{d} Q_j - \lambda_1 I \otimes \vec{U}_1 - \lambda_2 I \otimes \vec{U}_3$$

$$\Pi_{11} = \begin{bmatrix} \Psi & * & * & * & * \\ \bar{A}_1^T P \bar{\bar{A}} & \Psi_1 - Q_1 & * & * & * \\ \bar{A}_2^T P \bar{\bar{A}} & \bar{A}_2^T P \bar{A}_1 & \Psi_2 - Q_2 & * & * \\ \vdots & \vdots & \vdots & \ddots & * \\ \bar{A}_d^T P \bar{\bar{A}} & \bar{A}_d^T P \bar{A}_1 & \bar{A}_d^T P \bar{A}_d & \cdots & \Psi_d - Q_d \end{bmatrix}, \ \bar{\bar{A}} = \begin{bmatrix} A & 0 \\ 0 & A - \bar{\beta}_0 K^{(0)} C \end{bmatrix}$$

$$\Psi_i = \bar{A}_i^T P \bar{A}_i + \sigma_i^2 \tilde{A}_i^T P \tilde{A}_i, \ i = 1,2,\cdots d, \ \rho = \sqrt{\vartheta(1-\vartheta)}.$$

$$\Pi_{21} = \begin{bmatrix} \lambda_1 I \otimes \vec{U}_2^T & 0 \end{bmatrix}, \ \Pi_{31} = \begin{bmatrix} \lambda_2 I \otimes \vec{U}_4^T & 0 \end{bmatrix}$$

$$\Pi_{41} = \begin{bmatrix} \bar{F}^T P \bar{\bar{A}} & \bar{F}^T P \bar{A}_1 & \cdots & \bar{F}^T P \bar{A}_d \end{bmatrix}, \ \Pi_{44} = \bar{F}^T P \bar{F} + \rho^2 \tilde{F}^T P \tilde{F} - S,$$

$$\Pi_{51} = \begin{bmatrix} \bar{G}^T P \bar{\bar{A}} & \bar{G}^T P \bar{A}_1 & \cdots & \bar{G}^T P \bar{A}_d \end{bmatrix}, \ \Pi_{54} = \bar{G}^T P \bar{F} + \rho^2 \tilde{G}^T P \tilde{F},$$

$$\Pi_{55} = \bar{G}^T P \bar{G} + \rho^2 \tilde{G}^T P \tilde{G} - R.$$

Proof Construct the following Lyapunov-Krasovskii functional candidate as

$$V(\eta_k) = \sum_{i=1}^{6} V_i(\eta_k) \tag{4.11}$$

where

$$V_1(\eta_k) = \eta_k^T P \eta_k, \ V_2(\eta_k) = \sum_{j=1}^{d} \sum_{i=k-j}^{k-1} \eta_i^T Q_j \eta_i$$

$$V_3(\eta_k) = \sum_{i=k-\tau_k}^{k-1} \check{f}_i^T S \check{f}_i, \ V_4(\eta_k) = \sum_{j=k-\bar{\tau}+1}^{k-\underline{\tau}} \sum_{i=j}^{k-1} \check{f}_i^T S \check{f}_i,$$

$$V_5(\eta_k) = \sum_{i=k-\tau_k}^{k-1} \check{g}_i^T R \check{g}_i, \ V_6(\eta_k) = \sum_{j=k-\bar{h}+1}^{k-\underline{h}} \sum_{i=j}^{k-1} \check{g}_i^T R \check{g}_i.$$

Denote $\bar{\eta}_k = [\eta_k^T \quad \eta_{k-1}^T \quad \cdots \quad \eta_{k-d}^T]^{\mathrm{T}}$. Then calculating the difference of V_k along system (4.7) and taking the mathematical expectation, we have

$$E\{\Delta V_i(\eta_k)\} = E\{V_i(\eta_{k+1}) | \bar{\eta}_k\} - V_i(\eta_k) \tag{4.12}$$

Note that $E\{(\beta_k^{(i)} - \bar{\beta}_i)(\beta_k^{(j)} - \bar{\beta}_j)\} = 0$ when $i \neq j$, and $E\{(\beta_k^{(i)} - \bar{\beta}_i)(\vartheta_k - \bar{\vartheta})\} = 0$, so we have

$$E\{\Delta V_1\} = E\{\eta_{k+1}^T P \eta_{k+1} - \eta_k^T P \eta_k\} \tag{4.13}$$

$$= E\{\eta_k^T(\bar{A}^T P\bar{A} - P)\eta_k\} + \sum_{i=0}^{d} \eta_{k-i}^T \bar{A}_i^T P \bar{A}_i \,\eta_{k-i} + 2\sum_{i=0}^{d} \eta_{k-i}^T \bar{A}_i^T P\bar{A}\,\eta_k$$

$$+2\sum_{i=0}^{d-1}\sum_{j=i+1}^{d} \eta_{k-i}^T \bar{A}_i^T P\bar{A}_j\,\eta_{k-j} + \sum_{i=0}^{d} ((\beta_k^{(i)} - \bar{\beta}_i)(\vartheta_k - \bar{\vartheta}))^2\, \eta_{k-i}^T \tilde{A}_i^T P \tilde{A}_j \eta_{k-j}$$

$$+\breve{f}_{k-\tau_k}^T(\bar{F}^T P\bar{F} + (\vartheta_k - \bar{\vartheta})^2 \tilde{F}^T P\tilde{F})\breve{f}_{k-\tau_k} + 2\breve{f}_{k-\tau_k}^T \bar{F}^T P\bar{A}\eta_k$$

$$+\breve{g}_{k-h_k}^T(\bar{G}^T P\bar{G} + (\vartheta_k - \bar{\vartheta})^2 \tilde{G}^T P\tilde{G})\breve{g}_{k-h_k} + 2\breve{g}_{k-h_k}^T \bar{G}^T P\bar{A}\eta_k$$

$$+2\breve{f}_{k-\tau_k}^T(\bar{F}^T P\bar{G} + (\vartheta_k - \bar{\vartheta})^2 \tilde{F}^T P\tilde{G})\breve{g}_{k-h_k}$$

$$+2\sum_{i=0}^{d} \breve{f}^T(x_{k-\tau_k})\,\bar{F}^T P\bar{A}_i \eta_{k-i} + 2\sum_{i=0}^{d} \breve{g}^T(x_{k-h_k})\,\bar{G}^T P\bar{A}_i\eta_{k-i}\}$$

$$\mathrm{E}\{\Delta V_2\} = E\left\{\sum_{j=1}^{d} \eta_k^T Q_j \eta_k - \eta_{k-j}^T Q_j \eta_{k-j}\right\} \tag{4.14}$$

$$\mathrm{E}\{\Delta V_3\} \le E\left\{\breve{f}_k^T S\breve{f}_k - \breve{f}_{k-\tau_k}^T S\breve{f}_{k-\tau_k} + \sum_{i=k-\bar{\tau}+1}^{k-\underline{\tau}} \breve{f}_i^T S\breve{f}_i\right\} \tag{4.15}$$

$$\mathrm{E}\{\Delta V_4\} \le E\{(\bar{\tau} - \underline{\tau})\breve{f}_k^T S\breve{f}_k\} - E\left\{\sum_{i=k-\bar{\tau}+1}^{k-\underline{\tau}} \breve{f}_i^T S\breve{f}_i\right\} \tag{4.16}$$

$$\mathrm{E}\{\Delta V_5\} \le E\left\{\breve{g}_k^T R\breve{g}_k - \breve{g}_{k-\tau_k}^T R\breve{g}_{k-\tau_k} + \sum_{i=k-\bar{h}+1}^{k-\underline{h}} \breve{g}_i^T R\breve{g}_i\right\} \tag{4.17}$$

$$\mathrm{E}\{\Delta V_6\} \le E\{(\bar{h} - \underline{h})\breve{g}_k^T R\breve{g}_k\} - E\left\{\sum_{i=k-\bar{h}+1}^{k-\underline{h}} \breve{g}_i^T R\breve{g}_i\right\} \tag{4.18}$$

Furthermore, in terms of (4.12)-(4.18), we can obtain

$$E\{\Delta V(\eta_k)\} = \eta_k^T E\{V(\eta_{k+1})|\bar{\eta}_k\} - V(\eta_k) = \sum_{i=1}^{6} E\{\Delta V_i\} \le E\{\varsigma_k^T \bar{\Pi}\varsigma_k\} \tag{4.19}$$

where

$$\varsigma_k = \begin{bmatrix}\bar{\eta}_k^T & \breve{f}_k^T & \breve{g}_k^T & \breve{f}_{k-\tau_k}^T & \breve{g}_{k-h_k}^T\end{bmatrix}^T,$$

$$\bar{\Psi} = \bar{\bar{A}}^T P\bar{\bar{A}} + \sigma_0^2 \tilde{A}^T P\tilde{A} - P + \textstyle\sum_{j=1}^{d} Q_j,$$

$$\bar{\Pi}_{11} = \begin{bmatrix} \bar{\Psi} & * & * & * & * \\ \bar{A}_1^T P\bar{\bar{A}} & \Psi_1 - Q_1 & * & * & * \\ \bar{A}_2^T P\bar{\bar{A}} & \bar{A}_2^T P\bar{A}_1 & \Psi_2 - Q_2 & * & * \\ \vdots & \vdots & \vdots & \ddots & * \\ \bar{A}_d^T P\bar{\bar{A}} & \bar{A}_d^T P\bar{A}_1 & \bar{A}_d^T P\bar{A}_d & \cdots & \Psi_d - Q_d \end{bmatrix},$$

$$\bar{\Pi} = \begin{bmatrix} \bar{\Pi}_{11} & * & * & * & * \\ 0 & (\bar{\tau} - \underline{\tau} + 1)S & * & * & * \\ 0 & 0 & (\bar{h} - \underline{h} + 1)R & * & * \\ \Pi_{41} & 0 & 0 & \Pi_{44} & * \\ \Pi_{51} & 0 & 0 & \Pi_{54} & \Pi_{55} \end{bmatrix},$$

and the other parameters are defined by Theorem 4.1.

Note that from Assumptions 4.1 and 4.2, it follows that

$$[\breve{f}_k - (I \otimes U_1)\eta_k]^T[\breve{f}_k - (I \otimes U_2)\eta_k] \le 0 \tag{4.20}$$

$$[\breve{g}_k - (I \otimes U_3)\eta_k]^T[\breve{g}_k - (I \otimes U_4)\eta_k] \le 0 \tag{4.21}$$

One can immediately obtain

$$\begin{bmatrix}\eta_k\\ \breve{f}_k\end{bmatrix}^T\begin{bmatrix}I\otimes\vec{U}_1 & *\\ I\otimes\vec{U}_2^T & I\end{bmatrix}\begin{bmatrix}\eta_k\\ \breve{f}_k\end{bmatrix}\le 0 \tag{4.22}$$

$$\begin{bmatrix}\eta_k\\ \breve{g}_k\end{bmatrix}^T\begin{bmatrix}I\otimes\vec{U}_3 & *\\ I\otimes\vec{U}_4^T & I\end{bmatrix}\begin{bmatrix}\eta_k\\ \breve{g}_k\end{bmatrix}\le 0 \tag{4.23}$$

where

$$\vec{U}_1=(U_1^TU_2+U_2^TU_1)/2,\ \vec{U}_2=-(U_1^T+U_2^T)/2$$
$$\vec{U}_3=(U_3^TU_4+U_4^TU_3)/2,\ \vec{U}_4=-(U_3^T+U_4^T)/2$$

Subsequently, from (4.22) and (4.23), it follows that

$$\begin{aligned}E\{\Delta V(\eta_k)\}\le{}& \mathrm{E}\{\varsigma_k^T\bar{\Pi}\varsigma_k-\lambda_1\left[\breve{f}_k-(I\otimes U_1)\eta_k\right]^T\left[\breve{f}_k-(I\otimes U_2)\eta_k\right]\\ &-\lambda_2[\breve{g}_k-(I\otimes U_3)\eta_k]^T[\breve{g}_k-(I\otimes U_4)\eta_k]\}\le E\{\varsigma_k^T\Pi\varsigma_k\}\end{aligned} \tag{4.24}$$

A combination of (4.10) and (4.24) leads to

$$E\{\Delta V(\eta_k)\}\le\varsigma_k^T\Pi\varsigma_k<0 \tag{4.25}$$

It can be concluded from Lyapunov stability theory that the filtering error system (4.7) is asymptotically stable in mean square for given filter parameters. This proof is completed.

Theorem 4.2 Let the filter parameters $K^{(i)},\ i=0,1,\cdots d$ be given and γ be a prespecified positive constant. Then the filtering error system (4.7) is asymptotically stable in mean square and guarantees the H_∞ filtering performance γ, if there exist matrices $P>0,\ S>0,\ R>0,\ Q_j>0\ \ (j=1,2,\cdots d)$ and scalars $\lambda_1>0,\lambda_2>0$ such that

$$\Xi=\begin{bmatrix}\Xi_{11} & * & * & * & *\\ \Xi_{21} & \Xi_{22} & * & * & *\\ \Xi_{31} & 0 & \Xi_{33} & * & *\\ \Pi_{41} & \Xi_{42} & 0 & \Pi_{44} & *\\ \Pi_{51} & \Xi_{52} & 0 & \Pi_{54} & \Pi_{55}\end{bmatrix} \tag{4.26}$$

where

$$\Xi_{11}=\Pi_{11}+\bar{\bar{L}}^T\bar{\bar{L}},\ \bar{\bar{L}}=[\bar{L}\quad 0],\ \bar{\bar{\Gamma}}=\begin{bmatrix}\Gamma_1\\ \Gamma_1-\bar{\beta}_0K^{(0)}\Gamma_2\end{bmatrix},\ \Xi_{31}=\begin{bmatrix}\Pi_{21}\\ \Pi_{31}\end{bmatrix},$$

$$\Xi_{21}=\begin{bmatrix}\bar{\bar{\Gamma}}^TP\bar{\bar{A}}+\sigma_0^2\tilde{\Gamma}_0^TP\tilde{A}_0 & \bar{\bar{\Gamma}}^TP\bar{A}_1 & \bar{\bar{\Gamma}}^TP\bar{A}_2 & \cdots & \bar{\bar{\Gamma}}^TP\bar{A}_d\\ \bar{\Gamma}_1^TP\bar{\bar{A}} & M_1 & \bar{\Gamma}_1^TP\bar{A}_2 & \cdots & \bar{\Gamma}_1^TP\bar{A}_d\\ \bar{\Gamma}_2^TP\bar{\bar{A}} & \bar{A}_2^TP\bar{A}_1 & M_2 & \cdots & \bar{\Gamma}_2^TP\bar{A}_d\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \bar{\Gamma}_d^TP\bar{\bar{A}} & \bar{A}_d^TP\bar{A}_1 & \bar{A}_d^TP\bar{A}_2 & \cdots & M_d\end{bmatrix},$$

$$\mathrm{M}_i=\bar{\Gamma}_i^TP\bar{A}_i+\sigma_i^2\tilde{\Gamma}_i^TP\tilde{A}_i,\ \Theta_i=\bar{\Gamma}_i^TP\bar{\Gamma}_i+\sigma_i^2\tilde{\Gamma}_i^TP\tilde{\Gamma}_i,\ i=1,2,\cdots d$$

$$\Xi_{22}=\begin{bmatrix}\bar{\bar{\Gamma}}^TP\bar{\bar{\Gamma}}+\sigma_0^2\tilde{\Gamma}_0^TP\tilde{\Gamma}_0-\gamma^2I & * & * & * & *\\ \bar{\Gamma}_1^TP\bar{\bar{\Gamma}} & \Theta_1-\gamma^2I & \bar{\Gamma}_1^TP\bar{A}_2 & * & *\\ \bar{\Gamma}_2^TP\bar{\bar{\Gamma}} & \bar{\Gamma}_2^TP\bar{\Gamma}_1 & \Theta_2-\gamma^2I & * & *\\ \vdots & \vdots & \vdots & \ddots & *\\ \bar{\Gamma}_d^TP\bar{\bar{\Gamma}} & \bar{\Gamma}_d^TP\bar{\Gamma}_1 & \bar{\Gamma}_d^TP\bar{\Gamma}_2 & \cdots & \Theta_d-\gamma^2I\end{bmatrix}$$

$$\Xi_{33}=diag\{(\bar{\tau}-\underline{\tau}+1)S-\lambda_1I,\qquad (\bar{h}-\underline{h}+1)R-\lambda_2I\}$$

$$\Xi_{42} = \begin{bmatrix} \bar{F}^T P \bar{\bar{\Gamma}} & \bar{F}^T P \bar{\Gamma}_1 & \cdots & \bar{F}^T P \bar{\Gamma}_d \end{bmatrix}$$
$$\Xi_{52} = \begin{bmatrix} \bar{G}^T P \bar{\bar{\Gamma}} & \bar{G}^T P \bar{\Gamma}_1 & \cdots & \bar{G}^T P \bar{\Gamma}_d \end{bmatrix}$$

and the other parameters are defined by Theorem 4.1.

Proof First, it is obvious that $\Pi < 0$ under the condition that $\Xi < 0$. Consequently, according to Theorem 4.1, the filtering error system (4.7) with $w_k = 0$ is asymptotically stable in mean square. Now we are in a position to analyze the H_∞ performance for system (4.1) with all nonzero $w_k \in l_2[0, \infty)$.

To establish the H_∞ performance, we introduce the following performance index

$$J(n) = \sum_{k=0}^{\infty} E\{\tilde{z}_k^T \tilde{z}_k - \gamma^2 \sum_{i=0}^{d} w_{k-i}^T w_{k-i}\}. \tag{4.27}$$

Construct the same Lyapunov-Krasovskii functional as in Theorem 4.1. A similar manipulation as in the proof of Theorem 4.1 leads to

$$\begin{aligned} J(n) &= \sum_{k=0}^{\infty} E\left\{\tilde{z}_k^T \tilde{z}_k - \gamma^2 \sum_{i=0}^{d} w_{k-i}^T w_{k-i} + \Delta V(\eta_k)\right\} - E\{V(\eta_{k+1})\} \\ &\leq \sum_{k=0}^{\infty} E\left\{\eta_k^T \bar{L}^T \bar{L} \eta_k - \gamma^2 \sum_{i=0}^{d} w_{k-i}^T w_{k-i} + \Delta V(\eta_k)\right\} = \sum_{k=0}^{n} E\{\zeta_k^T \Xi \zeta_k\} \end{aligned} \tag{4.28}$$

where

$$\zeta_k = \begin{bmatrix} \bar{\eta}_k^T & \bar{w}_k^T & \check{f}_k^T & \check{g}_k^T & \check{f}_{k-\tau_k}^T & \check{g}_{k-h_k}^T \end{bmatrix}^T,$$
$$\bar{w}_k = [w_k^T \quad w_{k-1}^T \quad \cdots \quad w_{k-d}^T]^{\mathrm{T}}.$$

From (4.26), we have $J(n) \leq 0$. Letting $n \to \infty$, we obtain

$$\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\sum_{i=0}^{d} \|w_{k-i}\|^2\}.$$

which means that the desired H_∞ disturbance rejection attenuation level γ is achieved. The proof is completed.

4.4 H_∞ filter design

In this section, we focus on the filter design problem based on Theorem 4.2. A sufficient LMI condition for the existence of the proposed filter (4.5) is provided in the following theorem.

Theorem 4.3 For the discrete-time stochastic system (4.1) with possible consecutive packet dropouts, delayed measurements and randomly varying nonlinearities. There exists a filter in the form of (4.5) such that the filtering error system (4.7) is asymptotically stable in mean square under $w_k = 0$ and also satisfies the condition (4.9) under zero initial condition for any nonzero $w_k \in l_2[0, \infty)$, if there exist matrices $\tilde{P} > 0$, $\tilde{Q}_j > 0$ $(j = 1,2, \cdots d)$, $S > 0$, $R > 0$, W_j $(j = 1,2,3,4)$, $\hat{K}^{(i)}$,

$\widetilde{K}^{(i)}$ $(i = 0,1,\cdots d)$ and scalars $\lambda_1 > 0, \lambda_2 > 0$, such that the following LMI holds:

$$\begin{bmatrix} \Omega_{11} & * & * & * & * & * \\ 0 & \Omega_{22} & * & * & * & * \\ \Omega_{31} & 0 & \Omega_{33} & * & * & * \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & * & * \\ \Omega_{51} & \Omega_{52} & 0 & 0 & \Omega_{55} & * \\ 0 & 0 & \Omega_{63} & 0 & 0 & \Omega_{66} \end{bmatrix} < 0 \tag{4.29}$$

where

$$\Omega_{11} = \begin{bmatrix} \overline{\Omega}_{111} & 0 \\ 0 & \overline{\Omega}_{112} \end{bmatrix},\ \mathrm{V}_1 = \begin{bmatrix} I & I \\ I & 2I \end{bmatrix},\ \mathrm{V}_2 = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix},$$

$$\overline{\Omega}_{111} = -\tilde{P} + \sum_{j=1}^{d} \tilde{Q}_j + \lambda_1 \mathrm{V}_1 \otimes \vec{U}_1 + \lambda_2 \mathrm{V}_1 \otimes \vec{U}_3\ ,$$

$$\overline{\Omega}_{112} = diag\{\tilde{Q}_1,\ \tilde{Q}_2,\ \cdots, \tilde{Q}_d\},\ \Omega_{22} = -I_{d+1} \otimes \gamma^2 I_q,$$

$$\Omega_{31} = \begin{bmatrix} \lambda_1 \mathrm{V}_2 \otimes \vec{U}_2^T & 0 \\ \lambda_2 \mathrm{V}_2 \otimes \vec{U}_4^T & 0 \\ 0 & 0 \end{bmatrix},\ \Omega_{33} = diag\{\overline{\Omega}_{331},\ \overline{\Omega}_{332}, -S, -R\},$$

$$\overline{\Omega}_{331} = (\bar{\tau} - \underline{\tau} + 1)S - \lambda_1 I,\ \overline{\Omega}_{332} = (\bar{h} - \underline{h} + 1)R - \lambda_2 I,$$

$$\Omega_{41} = \begin{bmatrix} \overline{\Omega}_{410} & \overline{\Omega}_{411} & \overline{\Omega}_{412} & \cdots & \overline{\Omega}_{41d} \\ \bar{L} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\overline{\Omega}_{410} = \begin{bmatrix} W_1 A & W_1 A + W_2 A - \bar{\beta}_0 \widehat{K}^{(0)} C \\ (W_1 + W_3) A & \sum_{i=1}^{4} W_i A - \bar{\beta}_0 \widehat{K}^{(0)} C - \bar{\beta}_0 \widetilde{K}^{(0)} C \end{bmatrix},$$

$$\overline{\Omega}_{41j} = \begin{bmatrix} 0 & -\bar{\beta}_j \widehat{K}^{(j)} C \\ 0 & -\bar{\beta}_j \widehat{K}^{(j)} C - \bar{\beta}_j \widetilde{K}^{(j)} C \end{bmatrix},\ j = 1,2,\cdots d,$$

$$\Omega_{42} = \begin{bmatrix} \overline{\Omega}_{420} & \overline{\Omega}_{421} & \overline{\Omega}_{422} & \cdots & \overline{\Omega}_{42d} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\overline{\Omega}_{420} = \begin{bmatrix} W_1 \Gamma_1 + W_2 \Gamma_1 - \bar{\beta}_0 \widehat{K}^{(0)} \Gamma_2 \\ \sum_{i=1}^{4} W_i \Gamma_1 - \bar{\beta}_0 \widehat{K}^{(0)} \Gamma_2 - \bar{\beta}_0 \widetilde{K}^{(0)} \Gamma_2 \end{bmatrix},$$

$$\overline{\Omega}_{42j} = \begin{bmatrix} -\bar{\beta}_j \widehat{K}^{(j)} \Gamma_2 \\ -\bar{\beta}_j \widehat{K}^{(j)} \Gamma_2 - \bar{\beta}_j \widetilde{K}^{(j)} \Gamma_2 \end{bmatrix},\ j = 1,2,\cdots d,$$

$$\Omega_{43} = \begin{bmatrix} 0 & 0 & \overline{\Omega}_{431} & \overline{\Omega}_{432} \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \Omega_{44} = \begin{bmatrix} \tilde{P} - \widetilde{W} - \widetilde{W}^T & * \\ 0 & -I \end{bmatrix},$$

$$\overline{\Omega}_{431} = \begin{bmatrix} \bar{\vartheta} W_1 & \bar{\vartheta} W_2 \\ \bar{\vartheta}(W_1 + W_3) & \bar{\vartheta}(W_2 + W_4) \end{bmatrix},$$

$$\overline{\Omega}_{432} = \begin{bmatrix} (1 - \bar{\vartheta}) W_1 & (1 - \bar{\vartheta}) W_2 \\ (1 - \bar{\vartheta})(W_1 + W_3) & (1 - \bar{\vartheta})(W_2 + W_4) \end{bmatrix},$$

$$\Omega_{51} = diag\{\overline{\Omega}_{510}, \overline{\Omega}_{511}, \cdots, \overline{\Omega}_{51d}\},$$

$$\Omega_{52} = diag\{\overline{\Omega}_{520}, \overline{\Omega}_{521}, \cdots, \overline{\Omega}_{52d}\},$$

$$\Omega_{55} = I_{d+1} \otimes (\tilde{P} - \widetilde{W} - \widetilde{W}^T),$$

$$\overline{\Omega}_{51j} = \begin{bmatrix} -\sigma_j \widehat{K}^{(j)} C & -\sigma_j \widehat{K}^{(j)} C \\ -\sigma_j \widehat{K}^{(j)} C - \sigma_j \widetilde{K}^{(j)} C & -\sigma_j \widehat{K}^{(j)} C - \sigma_j \widetilde{K}^{(j)} C \end{bmatrix},$$

$$\overline{\Omega}_{52j} = \begin{bmatrix} -\sigma_j \widehat{K}^{(j)} \Gamma_2 \\ -\sigma_j \widehat{K}^{(j)} \Gamma_2 - \sigma_j \widetilde{K}^{(j)} \Gamma_2 \end{bmatrix},\ j = 0,1,\cdots d$$

$$\Omega_{63} = \begin{bmatrix} 0 & 0 & \overline{\Omega}_{631} & \overline{\Omega}_{632} \end{bmatrix},\ \Omega_{66} = \tilde{P} - \widetilde{W} - \widetilde{W}^T,$$

$$\bar{\Omega}_{631} = \begin{bmatrix} \rho(W_1 + W_2) & 0 \\ \rho\sum_{i=1}^{4} W_i & 0 \end{bmatrix},\ \bar{\Omega}_{632} = \begin{bmatrix} -\rho(W_1 + W_2) & 0 \\ -\rho\sum_{i=1}^{4} W_i & 0 \end{bmatrix}.$$

And other parameters are defined as in Theorem 4.1. Moreover, if the inequality (4.29) is feasible, the desired H_∞ filter in the form of (4.5) can be obtained as

$$K^{(i)} = (W_2^T W_2 + W_4^T W_4)^{-1}(W_2^T \hat{K}_i + W_4^T \tilde{K}_i),\ i = 0,1,\cdots d \tag{4.30}$$

Proof From Theorem 4.2, we know that there exists an admissible filter (4.5) such that the filtering error system (4.7) is asymptotically stable in mean square with a given H_∞ performance γ if there exist positive definite matrices $K^{(i)}$, $i = 0,1,\cdots d$ satisfying (4.26). By using Schur complement and Lemma 4.1, there exists a matrix W, such that

$$\begin{bmatrix} \Lambda_{11} & * & * & * & * & * \\ 0 & -I_{d+1}\otimes\gamma^2 I & * & * & * & * \\ \Lambda_{31} & 0 & \Lambda_{33} & * & * & * \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} & * & * \\ \Lambda_{51} & \Lambda_{52} & 0 & 0 & \Lambda_{55} & * \\ 0 & 0 & \Lambda_{63} & 0 & 0 & P - W - W^T \end{bmatrix} < 0 \tag{4.31}$$

with

$$\bar{\Lambda}_1 = -P + \sum_{j=1}^{d} Q_j + \lambda_1 \mathrm{V}_1 \otimes \vec{U}_1 + \lambda_2 \mathrm{V}_1 \otimes \vec{U}_3\ ,$$

$$\bar{\Lambda}_2 = diag\{Q_1,\ Q_2,\ \cdots, Q_d\},$$

$$\Lambda_{11} = \begin{bmatrix} \bar{\Lambda}_1 & 0 \\ 0 & \bar{\Lambda}_2 \end{bmatrix},\ \Lambda_{31} = \begin{bmatrix} \lambda_1 \mathrm{V}_2 \otimes \vec{U}_2^T & 0 \\ \lambda_2 \mathrm{V}_2 \otimes \vec{U}_4^T & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Lambda_{33} = \begin{bmatrix} (\bar{\tau} - \underline{\tau} + 1)S - \lambda_1 I & 0 & 0 & 0 \\ 0 & (\bar{h} - \underline{h} + 1)R - \lambda_2 I & 0 & 0 \\ 0 & 0 & -S & 0 \\ 0 & 0 & 0 & -R \end{bmatrix},$$

$$\Lambda_{41} = \begin{bmatrix} W\bar{\bar{A}} & W\bar{A}_1 & W\bar{A}_2 & \cdots & W\bar{A}_d \\ \bar{L} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Lambda_{42} = \begin{bmatrix} W\bar{\bar{\Gamma}} & W\bar{\Gamma}_1 & W\bar{\Gamma}_2 & \cdots & W\bar{\Gamma}_d \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\Lambda_{43} = \begin{bmatrix} 0 & 0 & W\bar{F} & W\bar{G} \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \Lambda_{44} = \begin{bmatrix} P - W - W^T & * \\ 0 & -I \end{bmatrix},$$

$$\Lambda_{51} = diag\{\sigma_0 W\tilde{A}_0, \sigma_1 W\tilde{A}_1, \cdots, \sigma_0 W\tilde{A}_d\},$$

$$\Lambda_{52} = diag\{\sigma_0 W\tilde{\Gamma}_0, \sigma_1 W\tilde{\Gamma}_1, \cdots, \sigma_0 W\tilde{\Gamma}_d\},$$

$$\Lambda_{55} = -I_{d+1} \otimes (P - W - W^T),$$

$$\Lambda_{63} = [0 \quad 0 \quad \rho W\tilde{F} \quad \rho W\tilde{G}].$$

Since P is positive definite, W is a nonsingular matrix. Without loss of generality, we partition W as $\mathrm{W} = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$ and introduce a matrix $\bar{\mathrm{W}} = \begin{bmatrix} X^{-1} & 0 \\ \bar{W}_{21} & V^{-1} \end{bmatrix}$, where $\bar{W}_{21}$ is uniquely determined from the equation:

$$YX^{-1} + V\bar{W}_{21} = I.$$

Define $= \begin{bmatrix} X^T & Y^T \\ 0 & V^T \end{bmatrix}$, then we have $J^T\bar{W} = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$.

Perform congruence transformation to (4.31) by $\tilde{J} = diag(\underbrace{\bar{W}^TJ,\cdots,\bar{W}^TJ}_{d+1},\underbrace{I,\cdots,I}_{d+5},\bar{W}^TJ,I,\underbrace{\bar{W}^TJ,\cdots,\bar{W}^TJ}_{d+2})$. and by defining $\tilde{P} = J^T\bar{W}P\bar{W}^TJ$, $\tilde{Q}_i = J^T\bar{W}Q_i\bar{W}^TJ$, $\widetilde{W} = J^T\bar{W}W\bar{W}^TJ$ and $\widehat{K}_i = W_2K^{(i)}$, $\widetilde{K}_i = W_4K^{(i)}$, $i = 0,1,\cdots d$, which leads to (4.29). Hence, if there exist matrices $\tilde{P} > 0$, $\tilde{Q}_j > 0$ $(j = 1,2,\cdots d)$, $S > 0$, $R > 0$, and matrices $\widehat{K}^{(i)}$, $\widetilde{K}^{(i)}$ $(i = 0,1,\cdots d)$, such that the LMI (4.29) is feasible, then the overall filtering error system is mean square stable and the constraint (4.9) is satisfied. Then from $\begin{bmatrix} W_2 \\ W_4 \end{bmatrix} K^{(i)} = \begin{bmatrix} \widehat{K}_i \\ \widetilde{K}_i \end{bmatrix}$ and noting that $\begin{bmatrix} W_2 \\ W_4 \end{bmatrix}$ is full column rank, we have (4.30) by the pseudo inverse. The proof is completed.

Furthermore, the minimal attenuation level γ can be obtained by solving the following problem

Problem 4.1 The optimal H_∞ filtering problem is

$$\min_{\substack{P>0,Q_j>0(j=1,2,\cdots,d),R>0,S>0,\lambda_1>0,\\ \lambda_2>0,W_i(i=1,2,3,4),\widehat{K}_j,\widetilde{K}_j(j=0,1,\cdots,d)}} \varpi$$

$$\text{subject to (4.29) with } \varpi = \gamma^2$$

The corresponding optimal H_∞ filtering performance level γ can be obtained by $\gamma = \sqrt{\varpi}$.

Remark 4.5 Up to now, we have developed the H_∞ filtering algorithm based on LMI technique. This LMI-based algorithm has a polynomial-time complexity, which is bounded by $O(\kappa N^3 \log(c/\varepsilon))$, where κ is the total row size of the LMI system, N is the total number of scalar decision variables, c is a data-dependent scaling factor, and ε is relative accuracy set for algorithm [28]. As for the H_∞ filtering problem we presented for the nonlinear system (4.1), the dimensions of the system variables are $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^r$ and $z_k \in \mathbb{R}^m$. From Theorem 4.3, we have $= 2n^2d + 3n^2 + nd + nrd + nr + 2n + 2$, $\kappa = 8n + 4nd + m$, therefore, the time complexity of our algorithm is $O(n^7d^4)$, which is obviously dependent on the maximal delay d and the dimension n of the state variable.

4.5 Simulation example

In this section, we present the F-404 aircraft engine system to demonstrate the applicability of the proposed filtering approach. By setting the sampling time as $T =$

$0.05\,s$, we obtain the following discretized nominal system matrix A and the measurement output matrix C as [29]:

$$A = \begin{bmatrix} 0.9270 & 0 & 0.1214 \\ 0.0082 & 0.9800 & -0.0189 \\ 0.0155 & 0 & 0.8885 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and the other matrices are set as

$$\Gamma_1 = [0.12 \quad 0.1 \quad 0.12]^T,\ \Gamma_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ L = [0.1 \quad 0.1 \quad -0.1].$$

$$\tau_k = 3 + (1 + (-1)^k/2),\ h_k = 2 + (1 + (-1)^k/2).$$

According to [30], virtually all aircraft engine systems are in some way disturbed by uncontrolled external forces which may enter the systems in many different ways, such as wind gusts, gravity gradients, structural vibrations, or sensor and actuator noise. These perturbations generally degrade the performance of the system and, in some situations, may even jeopardize the outcome of the engineering task.

To estimate the state of the F-404 aircraft engine system, the information needs to be transmitted from the aircraft in air to the control flat on the ground via wireless communication channels. In this case, when modeling the aircraft engine system, the randomly occurred nonlinear disturbances, the time delays and packet losses during the data transmission exist simultaneously. Our objective is to design a filter in the form of (4.5) such that the filtering error system (4.7) is asymptotically stable with a guaranteed H_∞ performance γ. Without loss of generality, we assume $\mathrm{d} = 2$, $\bar{\xi}_0 = 0.3$, $\bar{\xi}_1 = 0.5$, $\bar{\xi}_2 = 0.2$ and $\bar{\vartheta} = 0.4$.

Moreover, the nonlinear functions are given by

$$f(x) = [f_1(x) \quad f_2(x) \quad f_3(x)]^T,\ g(x) = [g_1(x) \quad g_2(x) \quad g_3(x)]^T,$$

$$\begin{cases} f_1(x) = \tanh(-x_1) + 0.3x_1 + 0.2x_2 + 0.1x_3 \\ f_2(x) = 0.1x_1 - \tanh(x_2) + 0.2x_2 + 0.3x_3 \\ f_3(x) = 0.1x_1 + 0.2x_3 - \tanh(x_3) \end{cases}$$

$$\begin{cases} g_1(x) = \tanh(-x_1) + 0.2x_1 + 0.1x_2 + 0.1x_3 \\ g_2(x) = 0.1x_1 - \tanh(x_2) + 0.2x_2 \\ g_3(x) = 0.1x_1 + 0.3x_3 - \tanh(x_3) \end{cases}$$

It is easy to verify that

$$U_1 = \begin{bmatrix} -0.7 & 0.2 & 0.1 \\ 0.1 & -0.8 & 0.3 \\ 0.1 & 0 & -0.8 \end{bmatrix},\ U_2 = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \\ 0.1 & 0 & 0.2 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} -0.8 & 0.1 & 0.1 \\ 0.1 & -0.8 & 0 \\ 0.1 & 0 & -0.8 \end{bmatrix},\ U_4 = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.2 \end{bmatrix},$$

$$\underline{\tau} = 3,\ \bar{\tau} = 4,\ \underline{h} = 2,\ \bar{h} = 3.$$

The H_∞ performance level is taken as $\gamma = 0.9$. With the above parameters and by using Matlab LMI toolbox to solve the feasibility of LMI in (4.29), we can obtain

$$K=\begin{bmatrix}0.2948 & 0.0519 & -0.1025 & -0.0267 & -0.1757 & -0.0446\\ 0.0347 & 0.2736 & -0.0384 & -0.1073 & -0.0864 & -0.2159\\ -0.0348 & -0.0499 & -0.0335 & -0.0155 & -0.0819 & -0.0356\end{bmatrix}$$

The data transmission case can be known from the random variables $\xi_k^{(0)}$, $\xi_k^{(1)}$ and $\xi_k^{(2)}$. For example, we describe the data transmission case in Table 4.1 when k from 1 to 10. From Table 4.1, we can see that y_3 and y_4 are lost, y_5 and y_8 are delayed by one step, y_7 is delayed by two steps and the other packets are received on time.

Let the disturbance $w_k = 2e^{-0.1k}\sin(0.1\pi k)$, the initial conditions are $x_0 = [-0.1 \quad 0.5 \quad 0.2]^T$ and $\hat{x}_0 = [0 \quad 0 \quad 0]^T$. The estimate of z_k and the estimation error are given as Figure 4.2 and Figure 4.3, from which it can be seen that the proposed filtering algorithm is applicable and effective. Moreover, by calculation, we can obtain

$$\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|^2\} / \sum_{k=0}^{\infty} E\{\sum_{i=0}^{2}\|w_{k-i}\|^2\} = 0.3094 < 0.81 = \gamma^2.$$

Hence, the effectiveness of H_∞ filter can be illustrated.

Table 4.1 Data Transmission in networks.

k	$\xi_k^{(0)}$	$\xi_k^{(1)}$	$\xi_k^{(2)}$	Y_k
1	1	1	1	$[y_1 \quad 0 \quad 0]^T$
2	1	1	1	$[y_2 \quad 0 \quad 0]^T$
3	0	0	0	$[0 \quad 0 \quad 0]^T$
4	0	0	0	$[0 \quad 0 \quad 0]^T$
5	0	0	0	$[0 \quad 0 \quad 0]^T$
6	1	1	0	$[y_6 \quad y_5 \quad 0]^T$
7	0	1	0	$[0 \quad 0 \quad 0]^T$
8	0	0	0	$[0 \quad 0 \quad 0]^T$
9	1	1	1	$[y_9 \quad y_8 \quad y_7]^T$
10	1	1	1	$[y_{10} \quad 0 \quad 0]^T$

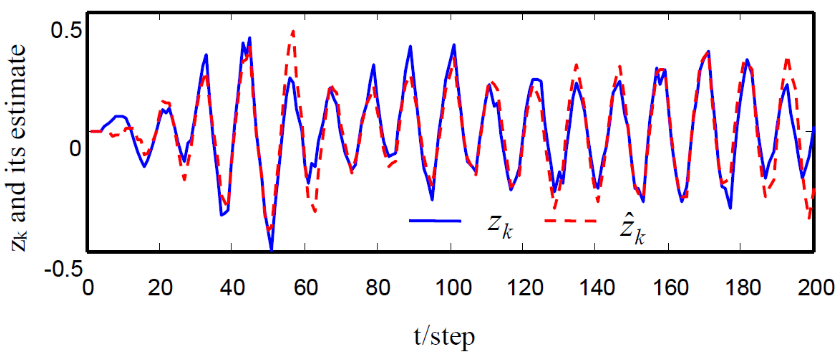

Figure 4.2 Estimated signal z_k and its estimate $\hat{z}_k$.

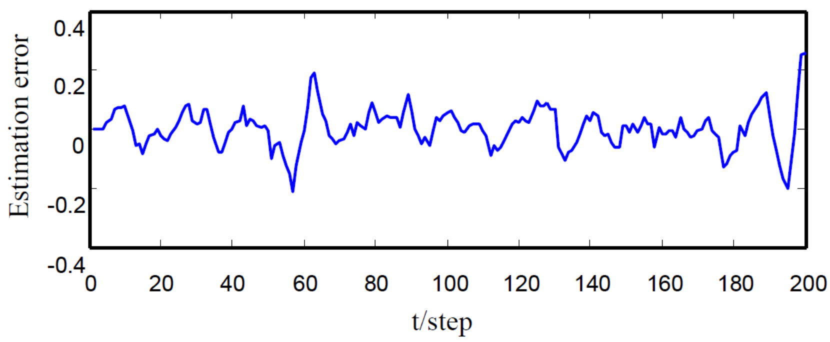

Figure 4.3 Estimation error of z_k.

Furthermore, in order to take a closer look at the effect of the largest delay number to the system performance, we give the comparison for different maximal transmission delay number in Table 4.2 under the case that the probabilities of the same random variable $\xi_k^{(i)}$ are same. For example, when $\mathrm{d} = 3$, the values of $\bar{\xi}_0$, $\bar{\xi}_1$, $\bar{\xi}_2$ are chosen as the same as the case of $\mathrm{d} = 2$, and the value $\bar{\xi}_3$ is selected arbitrary in $(0,1]$. Under such circumstance, the rates of a packet received on time, one-step delay and two-step delay are the same in the two scenarios. However, because the packet can be received with three-step delay when $\mathrm{d} = 3$, the packet dropout rate is smaller than the case of $\mathrm{d} = 2$ according to Remark 4.3.

Table 4.2 Comparison for the different maximal delay.

The maximal delay d	The probabilities of random variables $\xi_k^{(i)}$	The packet dropout rate	$\frac{\sum_{k=0}^{\infty} E\{\|\tilde{z}_k\|^2\}}{\sum_{k=0}^{\infty} E\{\sum_{i=0}^{d}\|w_{k-i}\|^2\}}$
1	$\bar{\xi}_0 = 0.3,\ \bar{\xi}_1 = 0.5$	0.35	0.4032
2	$\bar{\xi}_0 = 0.3,\ \bar{\xi}_1 = 0.5,$ $\bar{\xi}_2 = 0.2$	0.28	0.3094
3	$\bar{\xi}_0 = 0.3,\ \bar{\xi}_1 = 0.5,$ $\bar{\xi}_2 = 0.2,\ \bar{\xi}_3 = 0.7$	0.084	0.2777
4	$\bar{\xi}_0 = 0.3,\ \bar{\xi}_1 = 0.5,$ $\bar{\xi}_2 = 0.2,\ \bar{\xi}_3 = 0.7$ $\bar{\xi}_4 = 0.4$	0.0504	0.2276
5	$\bar{\xi}_0 = 0.3,\ \bar{\xi}_1 = 0.5,$ $\bar{\xi}_2 = 0.2,\ \bar{\xi}_3 = 0.7,$ $\bar{\xi}_4 = 0.4,\ \bar{\xi}_5 = 0.1$	0.0454	0.2258

From Table 4.2, we can see that when the packet dropout rate becomes server (*i.e.*, the maximal delay number is smaller), the H_∞ performance of the filtering error system becomes worse. Moreover, when the maximal delay number is large enough, the effect is not obvious. So, in the actual engineering application, we can give a prespecified number as the admissible maximal delay, if a packet is with a delay longer than that number, we can discard the packet and treat it as a packet dropout.

4.6 Conclusions

In this chapter, we have investigated the H_∞ filtering problem for a class of discrete-time network-based nonlinear systems with random bounded transmission delays and consecutive packet losses. Moreover, the nonlinearities are sector-bounded and occur randomly with time-varying states. By employing a new Lyapunov functional and stochastic analysis theory, sufficient conditions are presented for the filtering error system to be asymptotically stable in mean square sense with a prescribed H_∞ performance. The desired H_∞ filtering parameters are designed in terms of LMI which can be easily found by the Matlab LMI toolbox. The F-404 aircraft engine system has been given to show the effectiveness and applicability of the proposed filtering design method.

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Chapter 5 Fuzzy H_∞ Filtering for Nonlinear Networked Systems Subject to Sensor Saturations

5.1 Introduction

With the rapid development of communication and computer technologies, networked control system (NCS) has been widely used in telemedicine, intelligent transportation, aerospace and other fields [1]. While the introduction of network brings convenience to the system, due to network congestion, channel attenuation and other reasons, there are many uncertainties in the data transmission process, such as packet loss, delay and disorder. When the delay is greater than a predetermined value, it can also be considered as packet loss. If the phenomenon of packet loss is handled improperly, it may eventually lead to a sharp decline in the overall performance of the system or even collapse, so how to overcome the adverse impact of packet loss has become a common concern of scholars [2-5]. At present, there are mainly four strategies to conduct the packet loss, one is zero input strategy [2], the other is to hold input strategy [3], the third is predictive compensation strategy [4], and the fourth is multi-packet compensation strategy [5,6]. The so-called multi-packet compensation refers that the data received by the receiver may be one or more, or none in a sampling period due to the existence of transmission delays. The model presented in reference [5] describes a system with one-step delay and packet loss, based on which, the filters, predictors and smoothers are designed. Literature [6] proposes a model to describe the multiple delays and packet dropouts and a filter with prescribed H_∞ performance is designed. However, literature [5] only deals with linear systems, while literature [6] considers nonlinear disturbance, but the system model is still linear.

Nonlinear objects are ubiquitous in control engineering. Takagi-Sugeno fuzzy model [7] can approximate the nonlinear function defined on the compact set with arbitrary precision. It has been successfully applied to represent complex system whose precise model is difficult to obtain due to lack of sufficient knowledge [8-10]. The output tracking control problem is studied in reference [8] based on sampled data for networked control systems described by T-S fuzzy model. Reference [9] proposes a compensation strategy for a class of T-S fuzzy nonlinear networked control systems with packet dropouts in sensor-to-controller and controller-to-actuator channels. In reference [10], a new matrix decoupling method is proposed based on time-delay decomposition technique to design filter for T-S fuzzy systems. However, in the above literatures, the Lyapunov functions are chosen independent of the fuzzy rules when analyzing the stability performance, which makes the results conservative.

In addition, because of the limitation of principle, manufacturing technology and safety, the sensors used in engineering cannot recognize or provide signals with too large amplitude, so the saturation characteristics of the sensors are produced [11-15]. The filtering and control problems of networked control systems with sensor saturations are studied in references [11-13] and [14,15], respectively. For some types of sensors, such as Hall sensors, frequent saturation phenomena can cause device damage, lose the significance of measurement, and can no longer be used for measurement. How to solve the problem of signal source when sensor fails due to sensor saturation should also be paid attention to.

Based on the above analysis, this chapter studies the design of filters for a T-S fuzzy system with sensor saturation. The main contributions are as follows: (1) multi-packet compensation strategy is adopted to overcome the adverse effects of data loss on system performance; (2) sensor redundancy is proposed to solve the problem of signal source caused by sensor saturation; (3) in order to reduce the conservativeness of design, the rule-basis-dependent Lyapunov function is used to analysis the system's stability, and the filter parameters in the form of linear matrix inequality (LMI) are obtained which makes the filtering error system asymptotically stable and satisfies the specified H_∞ performance index. The effectiveness of the algorithm is verified by simulation results.

5.2 Problem formulation

5.2.1 T-S fuzzy model for nonlinear plants

Consider the following nonlinear discrete-time system:

$$x_{k+1} = f(x_k) + Bw_k \tag{5.1}$$

Assumption 5.1 $f(x_k)$ is a smooth function, which satisfies $f(0) = 0$, where $x_k \in \mathbb{R}^n$ is the system state vector.

Assumption 5.2 The external disturbance input $w_k \in \mathbb{R}^q$ is the energy finite signal, *i.e.*, w_k belongs to $l_2[0,\infty)$.

Assumption 5.3 B is the known matrix, which is used to describe the effects of the disturbance on the states.

For nonlinear system (5.1), it is approximated by a T-S fuzzy model with p rules. The ith fuzzy rule is given as follows:

Rule $\boldsymbol{i}$ If θ_k^1 is W_{i1} and θ_k^2 is W_{i2}, ..., and θ_k^p is W_{ip}, Then

$$\begin{cases} x_{k+1} = A_i x_k + B_i w_k \\ z_k = L_i x_k \end{cases} \tag{5.2}$$

where $\theta_k = [\theta_k^1, \theta_k^2, \cdots, \theta_k^p]$ is the premise variable, $W_{ij}(i,j = 1,2,\cdots,p)$ is the fuzzy set, $z_k \in \mathbb{R}^m$ is the estimated signal, A_i, B_i, and L_i are known constant matrices with appropriate dimensions.

Remark 5.1 Here, the introduction of z_k is more general. When L_i is the identity matrix, the estimated state is just x_k; When L_i is the general vector, the estimated signal is the component of x_k or its linear combination.

By using a singleton fuzzifier, product fuzzy inference, and center-average defuzzifier, the T-S fuzzy system (5.2) can be inferred as follows:

$$\begin{cases} x_{k+1} = \sum_{i=1}^{p} h_i\,(A_i x_k + B_i w_k) \\ z_k = \sum_{i=1}^{p} h_i\, L_i x_k \end{cases} \tag{5.3}$$

where $h_i \triangleq h_i(\theta_k) = \frac{\mu_i(\theta_k)}{\sum_{j=1}^{p}\mu_j(\theta_k)}$, $\mu_i(\theta_k) = \prod_{j=1}^{p} W_{ij}(\theta_k^j)$, $W_{ij}(\theta_k^j)$ is the grade of membership of θ_k^j in W_{ij}. Obviously, we have $h_i \geq 0$, $\sum_{i=1}^{p} h_i = 1$.

Remark 5.2 There is no unified approach to choose membership function, so we can select membership function appropriately according to the need. If we want the choice is optimal, the intelligent optimized method such as particle swarm optimization, ant colony algorithm and so on can be used to optimize the parameters in the membership function.

5.2.2 Redundant scheme for saturated sensor

When the signal amplitude detected by the sensor is too large, the saturation characteristics of the sensor components, such as differential pressure sensor and Hall sensor, will be generated. If the sensor is saturated for a long time, it will be easily damaged, and the measurement results have lost significance. For this reason, we propose a redundancy strategy of sensors [16,17] to solve the problem of signal sources. At this time, the output $y_k \in \mathbb{R}^r$ can be described as follows:

$$y_k = \sum_{i=1}^{p} h_i \left[\delta_k g(C_{1i}x_k) + (1-\delta_k)\phi(C_{2i}x_k)\right] \tag{5.4}$$

where the Bernoulli-distributed random variable δ_k is used to represent the switch signal between the main sensor and the redundant sensor. The statistic characteristic of δ_k is $\text{Prob}\{\delta_k = 1\} = \bar{\delta}$, $\text{Prob}\{\delta_k = 0\} = 1 - \bar{\delta}$, that is, $\delta_k = 1$ means the main sensor is at work, while $\delta_k = 0$ means the main sensor is lose efficacy, and then the redundant sensor is at work. The sensor saturations are described by $g(C_{1i}x_k)$ and $\phi(C_{2i}x_k)$, which satisfy

$$(g(C_{1i}x_k) - M_1C_{1i}x_k)^T(g(C_{1i}x_k) - M_2C_{1i}x_k) \le 0 \tag{5.5}$$

$$(\phi(C_{2i}x_k) - \bar{M}_1C_{2i}x_k)^T(\phi(C_{2i}x_k) - \bar{M}_2C_{2i}x_k) \le 0 \tag{5.6}$$

Here, $M_1, M_2 (M_2 > M_1 \ge 0)$ and $\bar{M}_1, \bar{M}_2 (\bar{M}_2 > \bar{M}_1 \ge 0)$ are constant matrices.

For the convenience of subsequent analysis, the nonlinear functions $g(C_{1i}x_k)$ and $\phi(C_{2i}x_k)$ are divided into a linear part and a nonlinear part [13], *i.e.*,

$$g(C_{1i}x_k) = g_n(C_{1i}x_k) + M_1C_{1i}x_k \tag{5.7}$$

$$\phi(C_{2i}x_k) = \phi_n(C_{2i}x_k) + \bar{M}_1C_{2i}x_k \tag{5.8}$$

where the nonlinear parts $g_n(C_{1i}x_k)$ and $\phi_n(C_{2i}x_k)$ are belonging to the sets Φ_n and Ω_n:

$$\Phi_n = \{g_n(C_{1i}x_k): g_n^T(C_{1i}x_k)(g_n(C_{1i}x_k) - \mathrm{M}C_{1i}x_k) \le 0\} \tag{5.9}$$

$$\Omega_n = \{\phi_n(C_{2i}x_k): \phi_n^T(C_{2i}x_k)(\phi_n(C_{2i}x_k) - \bar{\mathrm{M}}C_{2i}x_k) \le 0\} \tag{5.10}$$

with $M = M_2 - M_1 > 0$ and $\bar{M} = \bar{M}_2 - \bar{M}_1 > 0$.

5.2.3 Description of network induced phenomena

Assumption 5.4 The sensor is time-triggered.

Assumption 5.5 The buffer is equipped at the receiver side.

In the network environment, whether TCP or UDP protocol is adopted, the data will inevitably be delayed or lost in the transmission process [18]. Because of the existence of random time delay, the data received by the receiver may be one or more or none in a sampling period. According to reference [19], it can be described by the following model:

$$\tilde{y}_k = \xi_k^{(0)}y_k + \left(1 - \xi_{k-1}^{(0)}\right)\xi_k^{(1)}y_{k-1} + \cdots + \prod_{i=0}^{d-1}\left(1 - \xi_{k-d+i}^{(i)}\right)\xi_k^{(d)}y_{k-d} + D_iv_k \tag{5.11}$$

where $v_k \in \mathbb{R}^q$, belonging to $l_2[0,\infty)$, is the measurement noise. d is the maximum time delay. $\xi_k^{(i)}(i=0,1,\cdots,d)$ are the mutually independent random variables with Bernoulli distribution and satisfy $\text{Prob}\left\{\xi_k^{(i)}=1\right\}=\bar{\xi}_i$, $\text{Prob}\left\{\xi_k^{(i)}=0\right\}=1-\bar{\xi}_i$.

Remark 5.3 When $\xi_k^{(i)}(i=0,1,\cdots,d)$ are selected as different values, the data loss and d-step delay at the receiver can be described at the same time by using model (5.11). It can be known that the probability of data received on time is $\bar{\xi}_0$, the d-step delay rate is $\prod_{i=0}^{d-1}\left(1-\bar{\xi}_i\right)\bar{\xi}_d$, and the packet dropout rate is $1-\bar{\xi}_0-\prod_{i=0}^{d-1}\left(1-\bar{\xi}_i\right)\bar{\xi}_d$.

5.2.4 Design of fuzzy filter

For the nonlinear system (5.1) represented by T-S fuzzy model (5.2), by using parallel distributed compensation technique, we design the full-order filter in the following form:

Rule $\boldsymbol{i}$ If θ_k^1 is W_{i1} and θ_k^2 is W_{i2},…, and θ_k^p is W_{ip}, then

$$\begin{cases}\hat{x}_{k+1}=A_{fi}\hat{x}_k+B_{fi}\tilde{y}_k\\ \hat{z}_k=C_{fi}\hat{x}_k\end{cases} \tag{5.12}$$

where $\hat{x}_k\in\mathbb{R}^n$ is the estimated state, $\hat{z}_k\in\mathbb{R}^m$ is the output of filter, A_{fi} B_{fi}, and C_{fi} are the filter parameters to be designed.

Similarly, we can obtain the global model of the filter as follows:

$$\begin{cases}\hat{x}_{k+1}=\sum_{i=1}^{p}h_i\left(A_{fi}\hat{x}_k+B_{fi}\tilde{y}_k\right)\\ \hat{z}_k=\sum_{i=1}^{p}h_i\,C_{fi}\hat{x}_k\end{cases} \tag{5.13}$$

Remark 5.4 The state augment method can be used here to design the filter, that is, the augmented state $X_k=[x_k^T\quad x_{k-1}^T\quad\cdots\quad x_{k-d}^T]^T$ is introduced and then the filter $\hat{X}_k$ is designed for X_k. Obviously, the state augment method can lead to the increased dimension. The full-order filter we adopted in this chapter can avoid the dimension increasing.

Define $\xi_k^{(0)}=\alpha_k^{(0)}$, $\prod_{i=0}^{d-1}\left(1-\xi_{k-d+i}^{(i)}\right)\xi_k^{(i)}=\alpha_k^{(i)}$, $\alpha_k^{(i)}\delta_{k-i}=\sigma_k^{(i)}$, $i=0,1,\cdots,d$. Then, we have the following statistic characteristics:

$$E\left\{\sigma_k^{(i)}\right\}=\text{Prob}\left\{\sigma_k^{(i)}=1\right\}=\bar{\alpha}_i\bar{\delta}=\bar{\sigma}_i,$$

$$E\left\{\alpha_k^{(i)}\right\}=\text{Prob}\left\{\alpha_k^{(i)}=1\right\}=\bar{\alpha}_i,$$

$$E\left\{(\sigma_k^{(i)}-\bar{\sigma}_i)^2\right\}=\bar{\alpha}_i\bar{\delta}(1-\bar{\alpha}_i\bar{\delta})=\tilde{\rho}_i^2,$$

$$E\left\{(\alpha_k^{(i)}-\bar{\alpha}_i)^2\right\}=\bar{\alpha}_i(1-\bar{\alpha}_i)=\breve{\rho}_i^2,$$

$$E\left\{(\sigma_k^{(i)}-\bar{\sigma}_i)\right\}(\alpha_k^{(i)}-\bar{\alpha}_i)=\bar{\alpha}_i\bar{\delta}(1-\bar{\alpha}_i)=\bar{\rho}_i^2.$$

For description convenience, we let

$$\sum_{a_1,a_2,\cdots,a_s=1}^{p} h_{a_1} h_{a_2}\cdots h_{a_s}=\sum_{a_1=1}^{p} h_{a_1}\sum_{a_2=1}^{p} h_{a_2}\cdots\sum_{a_s=1}^{p} h_{a_s}\ (\forall s\geq 1).$$

From (5.2)-(5.4), (5.6) and (5.7), we can obtain the filtering error dynamic as follows:

$$\begin{cases} \varsigma_{k+1}=\sum_{i,j=1}^{p} h_i h_j\big[\sum_{l=0}^{d}(\bar{A}_{ij}^{(l)}+\tilde{\sigma}_{l,k}\tilde{A}_{ij}^{(l)}+\tilde{\alpha}_{l,k}\breve{A}_{ij}^{(l)})\varsigma_{k-l} \\ +\sum_{l=0}^{d}\big(\bar{B}_{ij}^{(l)}+\tilde{\sigma}_{l,k}\tilde{B}_{ij}^{(l)}+\tilde{\alpha}_{l,k}\breve{B}_{ij}^{(l)}\big)\bar{g}_n\big(H_{ij}\varsigma_{k-l}\big)+\bar{D}_{ij}\tilde{w}_k\big] \\ e_k=\sum_{i,j=1}^{p} h_i h_j\bar{L}_{ij}\varsigma_k \end{cases} \tag{5.14}$$

where

$$\varsigma_k=[x_k^T\quad \hat{x}_k^T]^T,\ e_k=z_k-\hat{z}_k$$

$$\bar{A}_{ij}^{(0)}=\begin{bmatrix} A_i & 0 \\ \bar{\sigma}_0 B_{fj}^{(0)}(M_1C_{1i}-\bar{M}_1C_{2i})+\bar{\alpha}_0 B_{fj}^{(0)}\bar{M}_1C_{2i} & A_{fj}\end{bmatrix},$$

$$\bar{A}_{ij}^{(l)}=\begin{bmatrix} 0 & 0 \\ \bar{\sigma}_l B_{fj}^{(l)}(M_1C_{1i}-\bar{M}_1C_{2i})+\bar{\alpha}_l B_{fj}^{(l)}\bar{M}_1C_{2i} & 0\end{bmatrix},\ (l=1,\cdots,d)$$

$$\tilde{A}_{ij}^{(l)}=\begin{bmatrix} 0 & 0 \\ B_{fj}^{(l)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\end{bmatrix},\ \breve{A}_{ij}^{(l)}=\begin{bmatrix} 0 & 0 \\ B_{fj}^{(l)}\bar{M}_1C_{2i} & 0\end{bmatrix},$$

$$\bar{B}_{ij}^{(l)}=\begin{bmatrix} 0 & 0 \\ \bar{\sigma}_d B_{fj}^{(l)} & (\bar{\alpha}_d-\bar{\sigma}_d)B_{fj}^{(l)}\end{bmatrix},\ \tilde{B}_{ij}^{(l)}=\begin{bmatrix} 0 & 0 \\ B_{fj}^{(l)} & -B_{fj}^{(l)}\end{bmatrix},$$

$$\breve{B}_{ij}^{(l)}=\begin{bmatrix} 0 & 0 \\ 0 & B_{fj}^{(l)}\end{bmatrix},\ \bar{g}_n\big(H_{ij}\varsigma_{k-l}\big)=\begin{bmatrix} g_n(H_{1i}\varsigma_{k-l}) \\ \phi_n(H_{2i}\varsigma_{k-l})\end{bmatrix}(l=0,1,\cdots,d)$$

$$\bar{D}_{ij}=\begin{bmatrix} B_i & 0 \\ 0 & B_{fj}D_i\end{bmatrix},\ \tilde{w}_k=[w_k^T\quad v_k^T]^T,\ H_{1i}=[C_{1i}\quad 0],\ H_{2i}=[C_{2i}\quad 0],\ \bar{L}_{ij}=[L_i\quad -C_{fj}].$$

5.2.5 Problem formulation

For the nonlinear system (5.1), based on the description of T-S fuzzy model (5.2), considering the saturation characteristics of the sensor, random packet loss and multi-step delay when the observed data arrived at the receiver side, the filter parameters A_{fi} B_{fi}, and C_{fi} in the form of (5.12) are designed to make the filtering error system (5.14) satisfy the following two conditions:

(i) (asymptotic stability) For any initial condition $\varsigma_0\in\mathbb{R}^{2n}$, when $\tilde{w}_k=0$, the filtering error system (5.14) is asymptotically stable, if the following condition holds:

$$\lim_{k\to\infty}E\{||\varsigma_k||^2\}=0 \tag{5.15}$$

(ii) (H_∞ performance) Under zero condition, for any non-zero $\tilde{w}_k\in l_2[0,\infty)$ and a prescribed scalar $\gamma>0$, the filtering error e_k satisfy the following H_∞ norm index:

$$\sum_{k=0}^{\infty}E\{||e_k||^2\}<\gamma^2\sum_{k=0}^{\infty}E\{||\tilde{w}_k||^2\} \tag{5.16}$$

Lemma 5.1 [20] For symmetric positive definite matrix S and any real matrices X_{ij} with suitable dimension, we have

$$\sum_{i,j,s,t=1}^{p} h_i\, h_j h_s h_t X_{ij}^T S X_{st} \leq \sum_{i,j=1}^{p} h_i\, h_j X_{ij}^T S X_{ij}$$

5.3 Filtering performance analysis

Theorem 5.1 Consider the nonlinear discrete-time system (5.1) represented by T-S fuzzy model (5.2). Assuming that the filter parameters A_{fi}, B_{fi}, and C_{fi} are known, the filtering error system (5.14) is asymptotically stable in the mean square and when $\widetilde{w}_k \neq 0$, the system (5.14) satisfies the prescribed H_∞ performance index γ, if there exist fuzzy-rule-dependent positive definite matrices $P_q > 0$, $Q_j > 0$ $(q = j = 1,2,\cdots,d)$ and scalars $\varepsilon_i > 0$ $(i = 0,1,\cdots,2d+1)$, such that the following matrix inequalities hold:

$$Z_{ij}^T \hat{P}_q Z_{ij} + \sum_{d=1}^{2} \bar{Z}_{ij}^{(d)T} \tilde{P}_q \bar{Z}_{ij}^{(d)} + \sum_{d=1}^{2} \bar{\bar{Z}}_{ij}^{(d)T} \vec{P}_q \bar{\bar{Z}}_{ij}^{(d)} + \hat{Z}_{ij}^T \hat{Z}_{ij} + \bar{P}_{ii} < 0 \tag{5.17}$$

$$\begin{aligned}&\left(Z_{ij} + Z_{ji}\right)^T \hat{P}_q \left(Z_{ij} + Z_{ji}\right) + \textstyle\sum_{d=1}^{2} (\bar{Z}_{ij}^{(d)} + \bar{Z}_{ji}^{(d)})^T \tilde{P}_q (\bar{Z}_{ij}^{(d)} + \bar{Z}_{ji}^{(d)}) + \sum_{d=1}^{2} (\bar{\bar{Z}}_{ij}^{(d)} + \bar{\bar{Z}}_{ji}^{(d)})\, \vec{P}_q (\bar{\bar{Z}}_{ij}^{(d)} + \\ &\bar{\bar{Z}}_{ji}^{(d)}) + (\hat{Z}_{ij} + \hat{Z}_{ji})^T (\hat{Z}_{ij} + \hat{Z}_{ji}) + 2(\bar{P}_{ii} + \bar{P}_{jj}) < 0 \quad (1 \leq i < j = q \leq p)\end{aligned} \tag{5.18}$$

where

$$Z_{ij} = \begin{bmatrix} \bar{A}_{ij}^{(0)} & \cdots & \bar{A}_{ij}^{(d)} & \bar{B}_{ij}^{(0)} & \cdots & \bar{B}_{ij}^{(d)} & \bar{D}_{ij} \\ \bar{L}_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{Z}_{ij}^{(1)} = \begin{bmatrix} \tilde{\rho}_0 \tilde{A}_{ij}^{(0)} & 0 & 0 & \tilde{\rho}_0 \tilde{B}_{ij}^{(0)} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \tilde{\rho}_d \tilde{A}_{ij}^{(d)} & 0 & 0 & \tilde{\rho}_d \tilde{B}_{ij}^{(d)} & 0 \end{bmatrix},$$

$$\bar{Z}_{ij}^{(2)} = \begin{bmatrix} \breve{\rho}_0 \breve{A}_{ij}^{(0)} & 0 & 0 & \breve{\rho}_0 \breve{B}_{ij}^{(0)} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \breve{\rho}_d \breve{A}_{ij}^{(d)} & 0 & 0 & \breve{\rho}_d \breve{B}_{ij}^{(d)} & 0 \end{bmatrix},$$

$$\bar{\bar{Z}}_{ij}^{(1)} = \upsilon \begin{bmatrix} \bar{\rho}_0 \tilde{A}_{ij}^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho}_d \tilde{A}_{ij}^{(d)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\rho}_0 \tilde{B}_{ij}^{(0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\rho}_d \tilde{B}_{ij}^{(d)} & 0 \end{bmatrix},$$

$$\bar{\bar{Z}}_{ij}^{(2)} = \upsilon \begin{bmatrix} \bar{\rho}_0 \breve{A}_{ij}^{(0)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\rho}_d \breve{A}_{ij}^{(d)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\rho}_0 \breve{B}_{ij}^{(0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\rho}_d \breve{B}_{ij}^{(d)} & 0 \end{bmatrix},$$

$$\hat{Z}_{ij} = \begin{bmatrix} \varepsilon_0 M C_{1i} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_d M C_{1i} & 0 & 0 & 0 & 0 \\ \varepsilon_{d+1} \bar{M} C_{2i} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{2d+1} \bar{M} C_{2i} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{P}_q = diag\{P_q, I\},\ \tilde{P}_q = I_2 \otimes P_q,\ \vec{P}_q = I_4 \otimes P_q, \upsilon = \sqrt{2},$$

$$\breve{P}_{is} = diag\{-P_q + \textstyle\sum_{l=1}^{d} Q_j^{(l)}, -Q_j^{(1)}, \cdots, -Q_j^{(d)}, -\varepsilon_0 I, \cdots, -\varepsilon_{2d+1} I\},$$

$$\bar{P}_{is} = diag\{\breve{P}_{is}, -\gamma^2 I\},$$

$$\tilde{\rho}_d^2 = \bar{\alpha}_d \bar{\delta}(1 - \bar{\alpha}_d \bar{\delta}), \breve{\rho}_d^2 = \bar{\alpha}_d (1 - \bar{\alpha}_d), \bar{\rho}_d^2 = \bar{\alpha}_d \bar{\delta}(1 - \bar{\alpha}_d).$$

Proof Choose the following fuzzy-rule-dependent Lyapunov function:

$$V_{1k}(\varsigma_k) = \varsigma_k^T \left(\sum_{i=1}^{p} h_i P_i \right) \varsigma_k \tag{5.19}$$

$$V_{2k}(\varsigma_k) = \textstyle\sum_{l=1}^{d} \sum_{i=k-l}^{k-1} \varsigma_k^T \left(\sum_{j=1}^{p} h_j Q_j^{(l)} \right) \varsigma_k \tag{5.20}$$

Then

$$\begin{aligned} E\{\Delta V_{1k}\} &= E\left\{ \varsigma_{k+1}^T \left(\sum_{i=1}^{p} h_i^+ P_i \right) \varsigma_{k+1} - \varsigma_k^T \left(\sum_{i=1}^{p} h_i P_i \right) \varsigma_k \right\} \\ &= E\left\{ \sum_{i,j,q,s,t=1}^{p} h_i h_j h_q^+ h_s h_t \left[\sum_{l=0}^{d} \left(\bar{A}_{ij}^{(l)} + \tilde{\sigma}_{l,k} \tilde{A}_{ij}^{(l)} + \tilde{\alpha}_{l,k} \breve{A}_{ij}^{(l)} \right) \varsigma_{k-l} \right. \right. \\ &\quad + \sum_{l=0}^{d} \left(\bar{B}_{ij}^{(l)} + \tilde{\sigma}_{l,k} \tilde{B}_{ij}^{(l)} + \tilde{\alpha}_{l,k} \breve{B}_{ij}^{(l)} \right) g_n\left(H_{ij} \varsigma_{k-l}\right) \\ &\quad \left. + \bar{D}_{ij} \widetilde{w}_k \right]^T P_q \left[\sum_{l=0}^{d} \left(\bar{A}_{ij}^{(l)} + \tilde{\sigma}_{l,k} \tilde{A}_{ij}^{(l)} + \tilde{\alpha}_{l,k} \breve{A}_{ij}^{(l)} \right) \varsigma_{k-l} \right. \\ &\quad \left. + \sum_{l=0}^{d} \left(\bar{B}_{ij}^{(l)} + \tilde{\sigma}_{l,k} \tilde{B}_{ij}^{(l)} + \tilde{\alpha}_{l,k} \breve{B}_{ij}^{(l)} \right) g_n\left(H_{ij} \varsigma_{k-l}\right) + \bar{D}_{ij} \widetilde{w}_k \right] \\ &\quad \left. - \varsigma_k^T \left(\sum_{i=1}^{p} h_i P_i \right) \varsigma_k \right\} \end{aligned} \tag{5.21}$$

$$\begin{aligned} = \sum_{i,j,q,s,t=1}^{p} & h_i h_j h_q^+ h_s h_t [-\varsigma_k^T P_i \varsigma_k + \sum_{l=0}^{d} \varsigma_{k-l}^T \left(\bar{A}_{ij}^{(l)T} P_q \bar{A}_{st}^{(l)} + \tilde{\rho}_l^2 \tilde{A}_{ij}^{(l)T} P_q \tilde{A}_{st}^{(l)} \right. \\ & + \breve{\rho}_l^2 \breve{A}_{ij}^{(l)T} P_q \breve{A}_{st}^{(l)} + 2\bar{\rho}_l^2 \tilde{A}_{ij}^{(l)T} P_q \breve{A}_{st}^{(l)}) \varsigma_{k-l} + \sum_{l=0}^{d} g_n^T\left(H_{ij} \varsigma_{k-l}\right) \left(\bar{B}_{ij}^{(l)T} P_q \bar{B}_{st}^{(l)} \right. \\ & + \tilde{\rho}_l^2 \tilde{B}_{ij}^{(l)T} P_q \tilde{B}_{st}^{(l)} + \breve{\rho}_l^2 \breve{B}_{ij}^{(l)T} P_q \breve{B}_{st}^{(l)} + 2\bar{\rho}_l^2 \tilde{B}_{ij}^{(l)T} P_q \breve{B}_{st}^{(l)}) g_n\left(H_{ij} \varsigma_{k-l}\right) \\ & + 2 \sum_{\tau=0}^{d-1} \sum_{\iota=\tau+1}^{d} \varsigma_{k-\tau}^T \bar{A}_{ij}^{(\tau)T} P_q \bar{A}_{st}^{(\iota)} \varsigma_{k-\iota} \\ & + 2 \sum_{\tau=0}^{d-1} \sum_{\iota=\tau+1}^{d} g_n^T\left(H_{ij} \varsigma_{k-\tau}\right) \bar{B}_{ij}^{(\tau)T} P_q \bar{B}_{st}^{(\iota)} g_n\left(H_{ij} \varsigma_{k-\iota}\right) \end{aligned}$$

$$+2\sum_{\tau=0}^{d-1}\sum_{\iota=\tau+1}^{d}\varsigma_{k-\tau}^{T}\bar{A}_{ij}^{(\tau)T}P_q\,\bar{B}_{st}^{(\iota)}g_n\big(H_{ij}\varsigma_{k-\iota}\big)+2\sum_{l=0}^{d}\varsigma_{k-l}^{T}\,\bar{A}_{ij}^{(l)T}P_q\bar{D}_{st}\widetilde{w}_k$$

$$+\,2\sum_{\tau=0}^{d-1}\sum_{\iota=\tau+1}^{d}g_n^T\big(H_{ij}\varsigma_{k-\tau}\big)\bar{B}_{ij}^{(\tau)T}P_q\bar{A}_{st}^{(\iota)}\,\varsigma_{k-\iota}$$

$$+\,2\sum_{l=0}^{d}g_n^T\big(H_{ij}\varsigma_{k-l}\big)\,\bar{B}_{ij}^{(l)T}P_q\bar{D}_{st}\widetilde{w}_k+2\sum_{l=0}^{d}\varsigma_{k-l}^{T}\,\big(\bar{A}_{ij}^{(l)T}P_q\bar{B}_{st}^{(l)}$$

$$+\,\tilde{\rho}_l^2\tilde{A}_{ij}^{(l)T}P_q\tilde{B}_{st}^{(l)}+\breve{\rho}_l^2\breve{A}_{ij}^{(l)T}P_q\breve{B}_{st}^{(l)}+2\bar{\rho}_l^2\tilde{A}_{ij}^{(l)T}P_q\breve{B}_{st}^{(l)}$$

$$+\,2\bar{\rho}_l^2\breve{A}_{ij}^{(l)T}P_q\tilde{B}_{st}^{(l)}\big)g_n(H_{st}\varsigma_{k-l})+\widetilde{w}_k^T\bar{D}_{ij}^TP_q\bar{D}_{st}\widetilde{w}_k$$

$$E\{\Delta V_{2k}\}=E\left\{\sum_{l=0}^{d}\sum_{i=k+1-l}^{k}\varsigma_i^T\left(\sum_{j=1}^{p}h_j^+\,Q_j^{(l)}\right)\varsigma_i-\sum_{l=0}^{d}\sum_{i=k-l}^{k-1}\varsigma_i^T\left(\sum_{j=1}^{p}h_j\,Q_j^{(l)}\right)\varsigma_i\right\}$$
$$=\sum_{j=1}^{p}h_j^+\sum_{l=1}^{d}\big(\varsigma_k^TQ_j^{(l)}\varsigma_k-\varsigma_{k-l}^TQ_j^{(l)}\varsigma_{k-l}\big) \tag{5.22}$$

Considering the constraint (5.8) of sensor saturation, and according to the inequality $2a^Tb\le a^Ta+b^Tb$, we can obtain:

$$2g_n^T\big(H_{ij}\varsigma_{k-l}\big)\big(g_n\big(H_{ij}\varsigma_{k-l}\big)-MH_{ij}\varsigma_{k-l}\big)=2g_n^T\big(H_{ij}\varsigma_{k-l}\big)g_n\big(H_{ij}\varsigma_{k-l}\big)-$$
$$2g_n^T\big(H_{ij}\varsigma_{k-l}\big)MH_{ij}\varsigma_{k-l}\le g_n^T\big(H_{ij}\varsigma_{k-l}\big)g_n\big(H_{ij}\varsigma_{k-l}\big)-\varsigma_{k-l}^TH_{ij}^TM^TMH_{ij}\varsigma_{k-l}\le 0 \tag{5.23}$$
$$l=0,1,\cdots,d$$

Similarly, from the relation $2\phi_n^T(C_{2i}x_k)(\phi_n(C_{2i}x_k)-\bar{M}C_{2i}x_k)\le 0$, we have

$$\phi_n^T(C_{2i}x_k)\phi_n(C_{2i}x_k)-\varsigma_{k-l}^TH_{ij}^TM^TMH_{ij}\varsigma_{k-l}\le 0 \tag{5.24}$$

In addition, we apply the inequality relation $2a^Tb\le a^Ta+b^Tb$ to the terms $2\bar{\rho}_l^2\tilde{A}_{ij}^{(l)T}P_q\breve{A}_{st}^{(l)}$, $2\bar{\rho}_l^2\tilde{B}_{ij}^{(l)T}P_q\breve{B}_{st}^{(l)}$, $2\bar{\rho}_l^2\tilde{A}_{ij}^{(l)T}P_q\breve{B}_{st}^{(l)}$ and $2\bar{\rho}_l^2\breve{A}_{ij}^{(l)T}P_q\tilde{B}_{st}^{(l)}$.

For the notation convenience, we introduce the following variable definitions:

$$\tilde{\eta}_k=\begin{bmatrix}\varsigma_k^T & \cdots & \varsigma_{k-d}^T & \bar{g}_n^T\big(H_{ij}\varsigma_k\big) & \cdots & \bar{g}_n^T\big(H_{ij}\varsigma_{k-d}\big)\end{bmatrix}^T,$$

$$\eta_k=\begin{bmatrix}\varsigma_k^T & \cdots & \varsigma_{k-d}^T & \bar{g}_n^T\big(H_{ij}\varsigma_k\big) & \cdots & \bar{g}_n^T\big(H_{ij}\varsigma_{k-d}\big) & \widetilde{w}_k^T\end{bmatrix}^T,$$

$$\Gamma_{ij}=\begin{bmatrix}\bar{A}_{ij}^{(0)} & \cdots & \bar{A}_{ij}^{(d)} & \bar{B}_{ij}^{(0)} & \cdots & \bar{B}_{ij}^{(d)}\end{bmatrix},$$

$$\bar{\Gamma}_{ij}^{(1)}=\begin{bmatrix}\tilde{\rho}_0\tilde{A}_{ij}^{(0)} & 0 & 0 & \tilde{\rho}_0\tilde{B}_{ij}^{(0)} & 0 & 0\\ 0 & \ddots & 0 & 0 & \ddots & 0\\ 0 & 0 & \tilde{\rho}_d\tilde{A}_{ij}^{(d)} & 0 & 0 & \tilde{\rho}_d\tilde{B}_{ij}^{(d)}\end{bmatrix},$$

$$\bar{\Gamma}_{ij}^{(2)}=\begin{bmatrix}\breve{\rho}_0\breve{A}_{ij}^{(0)} & 0 & 0 & \breve{\rho}_0\breve{B}_{ij}^{(0)} & 0 & 0\\ 0 & \ddots & 0 & 0 & \ddots & 0\\ 0 & 0 & \breve{\rho}_d\breve{A}_{ij}^{(d)} & 0 & 0 & \breve{\rho}_d\breve{B}_{ij}^{(d)}\end{bmatrix},$$

$$\bar{\bar{\Gamma}}_{ij}^{(1)}=diag\Big\{\upsilon\bar{\rho}_0\tilde{A}_{ij}^{(0)},\cdots,\upsilon\bar{\rho}_d\tilde{A}_{ij}^{(d)},\upsilon\bar{\rho}_0\tilde{B}_{ij}^{(0)},\cdots,\upsilon\bar{\rho}_d\tilde{B}_{ij}^{(d)}\Big\},$$

$$\bar{\bar{\Gamma}}_{ij}^{(2)}=diag\Big\{\upsilon\bar{\rho}_0\breve{A}_{ij}^{(0)},\cdots,\upsilon\bar{\rho}_d\breve{A}_{ij}^{(d)},\upsilon\bar{\rho}_0\breve{B}_{ij}^{(0)},\cdots,\upsilon\bar{\rho}_d\breve{B}_{ij}^{(d)}\Big\},$$

$$\hat{\Gamma}_{ij}=\begin{bmatrix}\varepsilon_1 MH_{ij} & 0 & 0 & 0 & 0\\ 0 & \ddots & 0 & 0 & 0\\ 0 & 0 & \varepsilon_d MH_{ij} & 0 & 0\\ \varepsilon_{d+1}\bar{M}H_{ij} & 0 & 0 & 0 & 0\\ 0 & \ddots & 0 & 0 & 0\\ 0 & 0 & \varepsilon_{2d+1}\bar{M}H_{ij} & 0 & 0\end{bmatrix}$$

Firstly, it is proved that the filtering error system (5.14) is asymptotically stable when $\widetilde{w}_k=0$. We have

$$\begin{aligned}E\{\Delta V_k\}\leq E\Big\{&\sum_{i,j,q,s,t=1}^{p} h_i h_j h_q^+ h_s h_t \tilde{\eta}_k^T\Big(\Gamma_{ij}^T P_q \Gamma_{st}+\sum_{d=1}^{2}\bar{\Gamma}_{ij}^{(d)T}\tilde{P}_q\bar{\Gamma}_{st}^{(d)}+\sum_{d=1}^{2}\bar{\bar{\Gamma}}_{ij}^{(d)T}\vec{P}_q\bar{\bar{\Gamma}}_{st}^{(d)}\\ &+\hat{\Gamma}_{ij}^T\hat{\Gamma}_{st}+\breve{P}_{is}\Big)\tilde{\eta}_k\\ &\leq\sum_{i,j,q=1}^{p} h_i h_j h_q^+ \tilde{\eta}_k^T\Big(\Gamma_{ij}^T P_q\Gamma_{ij}+\sum_{d=1}^{2}\bar{\Gamma}_{ij}^{(d)T}\tilde{P}_q\bar{\Gamma}_{ij}^{(d)}+\sum_{d=1}^{2}\bar{\bar{\Gamma}}_{ij}^{(d)T}\vec{P}_q\bar{\bar{\Gamma}}_{ij}^{(d)}\\ &+\hat{\Gamma}_{ij}^T\hat{\Gamma}_{ij}+\breve{P}_{ii}\Big)\tilde{\eta}_k\\ &=\sum_{i,q=1}^{p} h_i^2 h_q^+ \tilde{\eta}_k^T\Big(\Gamma_{ii}^T P_q\Gamma_{ii}+\sum_{d=1}^{2}\bar{\Gamma}_{ii}^{(d)T}\tilde{P}_q\bar{\Gamma}_{ii}^{(d)}+\sum_{d=1}^{2}\bar{\bar{\Gamma}}_{ii}^{(d)T}\vec{P}_q\bar{\bar{\Gamma}}_{ii}^{(d)}\\ &+\hat{\Gamma}_{ii}^T\hat{\Gamma}_{ii}+\breve{P}_{ii}\Big)\tilde{\eta}_k+\frac{1}{2}\sum_{i,j,q=1,i<j}^{p} h_i h_j h_q^+ \tilde{\eta}_k^T\Big[\big(\Gamma_{ij}+\Gamma_{ji}\big)^T P_q\big(\Gamma_{ij}+\Gamma_{ji}\big)\\ &+\sum_{d=1}^{2}\big(\bar{\Gamma}_{ij}^{(d)}+\bar{\Gamma}_{ji}^{(d)}\big)^T\tilde{P}_q\big(\bar{\Gamma}_{ij}^{(d)}+\bar{\Gamma}_{ji}^{(d)}\big)\\ &+\sum_{d=1}^{2}\big(\bar{\bar{\Gamma}}_{ij}^{(d)}+\bar{\bar{\Gamma}}_{ji}^{(d)}\big)^T\vec{P}_q\big(\bar{\bar{\Gamma}}_{ij}^{(d)}+\bar{\bar{\Gamma}}_{ji}^{(d)}\big)+\big(\hat{\Gamma}_{ij}+\hat{\Gamma}_{ji}\big)^T\big(\hat{\Gamma}_{ij}+\hat{\Gamma}_{ji}\big)\\ &+2\big(\breve{P}_{ii}+\breve{P}_{jj}\big)\Big]\tilde{\eta}_k\end{aligned}$$

By using Schur complement, $E\{\Delta V_k\}<0$ holds if and only if (5.17) and (5.18) hold. According to Lyapunov stability theory, the filtering error system (5.14) is asymptotically stable in the mean square sense when $\widetilde{w}_k=0$. Next, in order to analyze the H_∞ performance of filtering error system (5.14), we introduce an object function as follows:

$$J_n=E\Big\{\sum_{k=0}^{n}\big(e_k^T e_k-\gamma^2\widetilde{w}_k^T\widetilde{w}_k\big)\Big\}$$

Obviously, in order to prove H_∞ performance index (5.15) holds, it is necessary to prove $J_n<0$ under zero initial conditions. So, we have

$$\begin{aligned}J_n=E\Big\{&\sum_{k=0}^{n}\big(e_k^T e_k-\gamma^2\widetilde{w}_k^T\widetilde{w}_k+\Delta V_k\big)\Big\}-E\{V_{n+1}\}\\ &\leq E\Big\{\sum_{i,j,q,s,t=1}^{p} h_i h_j h_q^+ h_s h_t \eta_k^T\Big(\mathrm{Z}_{ij}^T\hat{P}_q\mathrm{Z}_{st}+\sum_{d=1}^{2}\bar{\mathrm{Z}}_{ij}^{(d)T}\tilde{P}_q\bar{\mathrm{Z}}_{st}^{(d)}\\ &+\sum_{d=1}^{2}\bar{\bar{\mathrm{Z}}}_{ij}^{(d)T}\vec{P}_q\bar{\bar{\mathrm{Z}}}_{st}^{(d)}+\hat{\mathrm{Z}}_{ij}^T\hat{\mathrm{Z}}_{st}+\bar{P}_{is}\Big)\eta_k\\ &\leq\sum_{i,j,q=1}^{p} h_i h_j h_q^+ \eta_k^T\Big(\mathrm{Z}_{ij}^T\hat{P}_q\mathrm{Z}_{ij}+\sum_{d=1}^{2}\bar{\mathrm{Z}}_{ij}^{(d)T}\tilde{P}_q\bar{\mathrm{Z}}_{ij}^{(d)}+\sum_{d=1}^{2}\bar{\bar{\mathrm{Z}}}_{ij}^{(d)T}\vec{P}_q\bar{\bar{\mathrm{Z}}}_{ij}^{(d)}\\ &+\hat{\mathrm{Z}}_{ij}^T\hat{\mathrm{Z}}_{ij}+\bar{P}_{ii}\Big)\eta_k\end{aligned}\tag{5.25}$$

$$= \sum_{i,q=1}^{p} h_i^2 h_q^+ \eta_k^T (\mathrm{Z}_{ii}^T \hat{P}_q \mathrm{Z}_{ii} + \sum_{d=1}^{2} \bar{\mathrm{Z}}_{ii}^{(d)T} \tilde{P}_q \bar{\mathrm{Z}}_{ii}^{(d)} + \sum_{d=1}^{2} \bar{\bar{\mathrm{Z}}}_{ii}^{(d)T} \vec{P}_q \bar{\bar{\mathrm{Z}}}_{ii}^{(d)} + \hat{\mathrm{Z}}_{ii}^T \hat{\mathrm{Z}}_{ii} + \bar{P}_{ii}) \eta_k$$

$$+ \frac{1}{2} \sum_{i,j,q=1,i<j}^{p} h_i h_j h_q^+ \eta_k^T \left[(\mathrm{Z}_{ij} + \mathrm{Z}_{ji})^T \hat{P}_q (\mathrm{Z}_{ij} + \mathrm{Z}_{ji}) \right.$$

$$+ \sum_{d=1}^{2} \left(\bar{\mathrm{Z}}_{ij}^{(d)} + \bar{\mathrm{Z}}_{ji}^{(d)} \right)^T \tilde{P}_q \left(\bar{\mathrm{Z}}_{ij}^{(d)} + \bar{\mathrm{Z}}_{ji}^{(d)} \right)$$

$$+ \sum_{d=1}^{2} \left(\bar{\bar{\mathrm{Z}}}_{ij}^{(d)} + \bar{\bar{\mathrm{Z}}}_{ji}^{(d)} \right)^T \vec{P}_q \left(\bar{\bar{\mathrm{Z}}}_{ij}^{(d)} + \bar{\bar{\mathrm{Z}}}_{ji}^{(d)} \right) + \left(\hat{\mathrm{Z}}_{ij} + \hat{\mathrm{Z}}_{ji} \right)^T \left(\hat{\mathrm{Z}}_{ij} + \hat{\mathrm{Z}}_{ji} \right)$$

$$\left. + 2 \left(\bar{P}_{ii} + \bar{P}_{jj} \right) \right] \eta_k$$

From (5.17) and (5.18), we have $J_n < 0$. Let $n \to \infty$, it is obtained that $\sum_{k=0}^{\infty} E\{||e_k||^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{||\widetilde{w}_k||^2\}$. The proof is completed.

5.4 H_∞ filter design

Theorem 5.2 Consider the discrete-time nonlinear system (5.1) represented by T-S fuzzy model (5.2). under the presence of sensor saturation and randomly occurred packet dropouts and time delays, for the given $\gamma > 0$, there exists an H_∞ filter such that the filtering error system (5.14) is asymptotically stable in the mean square sense and when $\widetilde{w}_k \neq 0$, the H_∞ performance index (5.15) holds if there exist the fuzzy-rule-dependent matrices $P_q > 0$, $Q_j > 0 (q = j = 1,2,\cdots,p)$, matrices $R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_2 \end{bmatrix}$, $\tilde{A}_{fi}$ $\tilde{B}_{fi}$, $\tilde{C}_{fi}$ and scalars $\varepsilon_i > 0$ (i = 0,1,...,2d+1)

$$\begin{bmatrix} \breve{P}_{ii} & * & * & * \\ \Psi_{ii} & -\Phi & * & * \\ \bar{\Psi}_{ii} & 0 & -\bar{\Phi} & * \\ \hat{\Gamma}_{ii} & 0 & 0 & -\tilde{\Phi} \end{bmatrix} < 0 \tag{5.26}$$

$$\begin{bmatrix} 2(\breve{P}_{ii} + \breve{P}_{jj}) & * & * & * \\ \Psi_{ij} + \Psi_{ji} & -\Phi & * & * \\ \bar{\Psi}_{ij} + \bar{\Psi}_{ji} & 0 & -\bar{\Phi} & * \\ \hat{\Gamma}_{ij} + \hat{\Gamma}_{ji} & 0 & 0 & -\tilde{\Phi} \end{bmatrix} < 0 \tag{5.27}$$

where

$$\Psi_{ij} = \begin{bmatrix} \bar{\Psi}_{11ij} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\Psi}_{22ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Psi}_{33ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Psi}_{44ij} & 0 \\ \bar{\Psi}_{51ij} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\Psi}_{62ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Psi}_{73ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Psi}_{84ij} & 0 \end{bmatrix}$$

$$\Psi_{11ij}=\begin{bmatrix}R_1A_i+\bar{\sigma}_0\tilde{B}_{fj}^{(0)}(M_1C_{1i}-\bar{M}_1C_{2i})+\bar{\alpha}_0\tilde{B}_{fj}^{(0)}\bar{M}_1C_{2i} & \tilde{A}_{fj}\\ R_3A_i+\bar{\sigma}_0\tilde{B}_{fj}^{(0)}(M_1C_{1i}-\bar{M}_1C_{2i})+\bar{\alpha}_0\tilde{B}_{fj}^{(0)}\bar{M}_1C_{2i} & \tilde{A}_{fj}\end{bmatrix},$$

$$\Psi_{12ij}=\begin{bmatrix}\bar{\sigma}_d\tilde{B}_{fj}^{(d)}(M_1C_{1i}-\bar{M}_1C_{2i})+\bar{\alpha}_d\tilde{B}_{fj}^{(d)}\bar{M}_1C_{2i} & 0\\ \bar{\sigma}_d\tilde{B}_{fj}^{(d)}(M_1C_{1i}-\bar{M}_1C_{2i})+\bar{\alpha}_d\tilde{B}_{fj}^{(d)}\bar{M}_1C_{2i} & 0\end{bmatrix},$$

$$\Psi_{13ij}=\begin{bmatrix}\bar{\sigma}_0\tilde{B}_{fj}^{(0)} & (\bar{\alpha}_0-\bar{\sigma}_0)\tilde{B}_{fj}^{(0)}\\ \bar{\sigma}_0\tilde{B}_{fj}^{(0)} & (\bar{\alpha}_0-\bar{\sigma}_0)\tilde{B}_{fj}^{(0)}\end{bmatrix}, \Psi_{14ij}=\begin{bmatrix}\bar{\sigma}_d\tilde{B}_{fj}^{(d)} & (\bar{\alpha}_d-\bar{\sigma}_d)\tilde{B}_{fj}^{(d)}\\ \bar{\sigma}_d\tilde{B}_{fj}^{(d)} & (\bar{\alpha}_d-\bar{\sigma}_d)\tilde{B}_{fj}^{(d)}\end{bmatrix},$$

$$\Psi_{15ij}=\begin{bmatrix}R_1B_i & \tilde{B}_{fj}D_i\\ R_3B_i & \tilde{B}_{fj}D_i\end{bmatrix}, \Psi_{21ij}=\begin{bmatrix}L_i & -\tilde{C}_{fj}\end{bmatrix},$$

$$\Psi_{31ij}=\begin{bmatrix}\tilde{\rho}_0\tilde{B}_{fj}^{(0)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\\ \tilde{\rho}_0\tilde{B}_{fj}^{(0)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\end{bmatrix},\ \Psi_{33ij}=\begin{bmatrix}\tilde{\rho}_0\tilde{B}_{fj}^{(0)} & -\tilde{\rho}_0\tilde{B}_{fj}^{(0)}\\ \tilde{\rho}_0\tilde{B}_{fj}^{(0)} & -\tilde{\rho}_0\tilde{B}_{fj}^{(0)}\end{bmatrix},$$

$$\Psi_{44ij}=\begin{bmatrix}\tilde{\rho}_d\tilde{B}_{fj}^{(d)} & -\tilde{\rho}_d\tilde{B}_{fj}^{(d)}\\ \tilde{\rho}_d\tilde{B}_{fj}^{(d)} & -\tilde{\rho}_d\tilde{B}_{fj}^{(d)}\end{bmatrix}$$

$$\Psi_{42ij}=\begin{bmatrix}\tilde{\rho}_d\tilde{B}_{fj}^{(d)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\\ \tilde{\rho}_d\tilde{B}_{fj}^{(d)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\end{bmatrix}, \Psi_{51ij}=\begin{bmatrix}\breve{\rho}_0\tilde{B}_{fj}^{(0)}\bar{M}_1C_{2i} & 0\\ \breve{\rho}_0\tilde{B}_{fj}^{(0)}\bar{M}_1C_{2i} & 0\end{bmatrix}, \Psi_{53ij}=\begin{bmatrix}\breve{\rho}_0\tilde{B}_{fj}^{(0)} & \breve{\rho}_0\tilde{B}_{fj}^{(0)}\\ \breve{\rho}_0\tilde{B}_{fj}^{(0)} & \breve{\rho}_0\tilde{B}_{fj}^{(0)}\end{bmatrix},$$

$$\Psi_{62ij}=\begin{bmatrix}\breve{\rho}_d\tilde{B}_{fj}^{(d)}\bar{M}_1C_{2i} & 0\\ \breve{\rho}_d\tilde{B}_{fj}^{(d)}\bar{M}_1C_{2i} & 0\end{bmatrix},\ \Psi_{64ij}=\begin{bmatrix}\breve{\rho}_d\tilde{B}_{fj}^{(d)} & \breve{\rho}_d\tilde{B}_{fj}^{(d)}\\ \breve{\rho}_d\tilde{B}_{fj}^{(d)} & \breve{\rho}_d\tilde{B}_{fj}^{(d)}\end{bmatrix},$$

$$\bar{\Psi}_{11ij}=\upsilon\begin{bmatrix}\bar{\rho}_0\tilde{B}_{fj}^{(0)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\\ \bar{\rho}_0\tilde{B}_{fj}^{(0)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\end{bmatrix}, \bar{\Psi}_{22ij}=\upsilon\begin{bmatrix}\bar{\rho}_d\tilde{B}_{fj}^{(d)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\\ \bar{\rho}_d\tilde{B}_{fj}^{(d)}(M_1C_{1i}-\bar{M}_1C_{2i}) & 0\end{bmatrix},$$

$$\bar{\Psi}_{33ij}=\upsilon\begin{bmatrix}\bar{\rho}_0\tilde{B}_{fj}^{(0)} & -\bar{\rho}_0\tilde{B}_{fj}^{(0)}\\ \bar{\rho}_0\tilde{B}_{fj}^{(0)} & -\bar{\rho}_0\tilde{B}_{fj}^{(0)}\end{bmatrix}, \bar{\Psi}_{44ij}=\upsilon\begin{bmatrix}\bar{\rho}_d\tilde{B}_{fj}^{(d)} & -\bar{\rho}_d\tilde{B}_{fj}^{(d)}\\ \bar{\rho}_d\tilde{B}_{fj}^{(d)} & -\bar{\rho}_d\tilde{B}_{fj}^{(d)}\end{bmatrix},$$

$$\bar{\Psi}_{51ij}=\upsilon\begin{bmatrix}\bar{\rho}_0\tilde{B}_{fj}^{(0)}\bar{M}_1C_{2i} & 0\\ \bar{\rho}_0\tilde{B}_{fj}^{(0)}\bar{M}_1C_{2i} & 0\end{bmatrix}, \bar{\Psi}_{62ij}=\upsilon\begin{bmatrix}\bar{\rho}_d\tilde{B}_{fj}^{(d)}\bar{M}_1C_{2i} & 0\\ \bar{\rho}_d\tilde{B}_{fj}^{(d)}\bar{M}_1C_{2i} & 0\end{bmatrix},$$

$$\bar{\Psi}_{73ij}=\upsilon\begin{bmatrix}0 & \bar{\rho}_0\tilde{B}_{fj}^{(0)}\\ 0 & \bar{\rho}_0\tilde{B}_{fj}^{(0)}\end{bmatrix},\ \bar{\Psi}_{84ij}=\upsilon\begin{bmatrix}0 & \bar{\rho}_d\tilde{B}_{fj}^{(d)}\\ 0 & \bar{\rho}_d\tilde{B}_{fj}^{(d)}\end{bmatrix},$$

$$\Phi=\mathrm{diag}\Big\{P_q-R-R^T \quad I \quad \underbrace{P_q-R-R^T \quad \cdots \quad P_q-R-R^T}_{2(d+1)}\Big\},$$

$$\bar{\Phi}=\mathrm{diag}\left\{\underbrace{P_q-R-R^T \quad \cdots \quad P_q-R-R^T}_{4(d+1)}\right\},$$

$$\widetilde{\Phi}=\mathrm{diag}\{\varepsilon_0 I \quad \varepsilon_1 I \quad \cdots \quad \varepsilon_{2d+1} I\}.$$

If the inequalities (5.26) and (5.27) exist the feasible solution, then the filter parameters are given as follows:

$$A_{fj}=R_2^{-1}\tilde{A}_{fj},\ B_{fj}=R_2^{-1}\tilde{B}_{fj},\ C_{fj}=\tilde{C}_{fj} \tag{5.28}$$

Proof By using Schur complement, we can rewrite the matrix inequalities (5.17) and (5.18) in Theorem 5.1 as the following forms:

$$\begin{bmatrix}\breve{P}_{ii} & *\\ \Sigma_{ii} & -\Xi\end{bmatrix}<0 \tag{5.29}$$

$$\begin{bmatrix} 2(\breve{P}_{ii} + \breve{P}_{jj}) & * \\ \Sigma_{ij} + \Sigma_{ji} & -\Xi \end{bmatrix} < 0 \tag{5.30}$$

where

$$\Xi = \mathrm{diag}\left\{ P_q^{-1} \quad I \quad \underbrace{P_q^{-1} \quad \cdots \quad P_q^{-1}}_{6(d+1)} \quad \underbrace{I \quad \cdots \quad I}_{2(d+1)} \right\},$$
$$\Sigma_{ij} = \begin{bmatrix} Z_{ij}^T & \bar{Z}_{ij}^{(1)T} & \bar{Z}_{ij}^{(2)T} & \bar{\bar{Z}}_{ij}^{(1)T} & \bar{\bar{Z}}_{ij}^{(2)T} & \hat{Z}_{ij}^T \end{bmatrix}^T,$$

We introduce a matrix $R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_2 \end{bmatrix}$ and assume R_2 is nonsingular. Make the congruence transformation to inequalities (5.29) and (5.30) by $J = diag\left\{ \underbrace{I \quad \cdots \quad I}_{3(d+1)} \quad R \quad I \quad \underbrace{R \quad \cdots \quad R}_{6(d+1)} \quad \underbrace{I \quad \cdots \quad I}_{2(d+1)} \right\}$ and its transpose. Due to $(P_q - R)P_q^{-1}(P_q - R)^T \geq 0$, we have $-RP_q^{-1}R^T \leq P_q - R - R^T$. So, the Equations (5.31) and (5.32) hold.

$$\begin{bmatrix} \breve{P}_{ii} & * \\ \bar{R}\Sigma_{ii} & -\bar{\Xi} \end{bmatrix} < 0 \tag{5.31}$$

$$\begin{bmatrix} 2(\breve{P}_{ii} + \breve{P}_{jj}) & * \\ \bar{R}(\Sigma_{ij} + \Sigma_{ji}) & -\bar{\Xi} \end{bmatrix} < 0 \tag{5.32}$$

where

$$\bar{\Xi} = \mathrm{diag}\{-\Phi \quad -\bar{\Phi} \quad -\tilde{\Phi}\},$$
$$\bar{R} = \left\{ R \quad I \quad \underbrace{R \quad \cdots \quad R}_{6(d+1)} \quad \underbrace{I \quad \cdots \quad I}_{2(d+1)} \right\}.$$

Define $\tilde{A}_{fj} = R_2 A_{fj}$, $\tilde{B}_{fj} = R_2 B_{fj}$, $\tilde{C}_{fj} = C_{fj}$, we can obtain (5.26) and (5.27). Then, we have

$$A_{fj} = R_2^{-1}\tilde{A}_{fj},\ B_{fj} = R_2^{-1}\tilde{B}_{fj},\ C_{fj} = \tilde{C}_{fj}$$

that is (5.28) holds. The proof is completed.

Remark 5.5 From the inequalities (5.26) and (5.27), by solving the following convex optimization problems:

$$\begin{gathered} \min_{\substack{P_q>0, Q_j>0 (q=j=1,2,\cdots,p) \\ \tilde{A}_{fj}, \tilde{B}_{fj}, \tilde{C}_{fj}, \varepsilon_i>0 (i=0,\cdots,2d+1)}} \varpi \\ \text{s.t. (5.26), (5.27) with } \varpi = \gamma^2 \end{gathered} \tag{5.33}$$

we can obtain the optimal H_∞ performance index $\gamma^* = \sqrt{\varpi^*}$, where ϖ^* is the optimal value of ϖ.

Remark 5.6 In some circumstances, there are some modeling errors for nonlinear system (5.1) represented by T-S fuzzy model (5.2). Considering the approximation error, the nonlinear system (5.1) can be equivalent to the following system:

$$x_{k+1} = \sum_{i=1}^{p} h_i\,(A_i x_k + B_i w_k) + \Delta f_k \tag{5.34}$$

where $\Delta f_k = f(x_k) - \sum_{i=1}^{p} h_i\,(A_i x_k + B_i w_k)$ is the modeling error. How to deal with the modeling error is our future research topic.

5.5 Simulation example

In order to illustrate the effectiveness of the proposed algorithm, we consider a nonlinear inverted pendulum system in reference [21]:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{gsin\big(x_1(t)\big) + \left(\frac{2b}{lm}\right)x_2(t)}{\frac{4l}{3} - amlcos^2\big(x_1(t)\big)} + \frac{\left(\frac{aml}{2}\right)x_2^2(t)\sin\big(2x_1(t)\big)}{\frac{4l}{3} - amlcos^2\big(x_1(t)\big)} + 10w(t)$$

where $x_1(t)$ represents the vertical direction angle of inverted pendulum, $x_2(t)$ is the angular velocity, $g = 9.8\,m/s^2$ is the gravity acceleration, m is the mass of inverted pendulum, $a = 1/(m+M)$, M is the mass of cart, $2l$ is the length of inverted pendulum, b is the damping coefficient between the inverted pendulum and the shaft, $w(t)$ is the disturbance input. According to reference [21], let $m = 2$ kg, $M = 8\,kg$, $l = 0.5$ m and $b = 0.5\,N\cdot m/s$. The discrete-time T-S fuzzy model of the system can be obtained by linearizing the system at the origin and $x = (\pm 60°, 0)$ when the sampling period (T = 0.01 s) as follows:

$$A_1 = \begin{bmatrix} 0.9921 & 0.0098 \\ -0.1702 & 0.9748 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9927 & 0.0098 \\ -0.0577 & 0.9773 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$

$$C_{11} = [0.6 \quad 0.6], C_{12} = [0.6 \quad 0.6], C_{21} = [1 \quad 1], C_{22} = [1 \quad 1]$$

$$L_1 = [0 \quad 1], L_2 = [0 \quad 1], D_1 = 1,\ D_2 = 1,$$

In the simulation, it is assumed that the maximum delay is $d = 2$. Let $\bar{\xi}_0 = 0.3$, $\bar{\xi}_1 = 0.2$, $\bar{\delta} = 0.6$, that is, the probability of receiving observation data on time is 0.3, the probability of receiving with one-step delay is 0.14, and the probability of losing is 0.56. Table 5.1 illustrates the data received by the receiver when the different values of $\xi_k^{(i)} (i = 0,1)$ are taken. We can see from Table 5.1 that when $k = 1, 2, 6$, the data is received on time, when $k = 3$, the data is lost, when $k = 4$, the data is received with one-time delay, and when $k = 5$, the two data is received in the meanwhile at the receiver side.

Table 5.1 An example of receiving data at the receiver side.

k	1	2	3	4	5	6
$\xi_k^{(0)}$	1	1	0	0	1	1
$\xi_k^{(1)}$	0	1	0	1	1	1
$\tilde{y}_k$	y_1	y_2	0	y_3	y_4+y_5	y_6

By solving the problem of (5.33), we can obtain the optimal interference suppression performance $\gamma^* = 5.7774$ and the filter parameters are as follows:

$$A_{f1} = \begin{bmatrix} 0.9834 & 0.0088 \\ -0.1658 & 0.9648 \end{bmatrix},\ A_{f2} = \begin{bmatrix} 0.9896 & 0.0103 \\ -0.0585 & 0.9653 \end{bmatrix},$$

$$B_{f1} = \begin{bmatrix} -0.0012 \\ -0.0130 \end{bmatrix},\ B_{f2} = \begin{bmatrix} -0.0026 \\ -0.0120 \end{bmatrix},$$

$$C_{f1} = [0.0252 \quad -1.0167],\ C_{f2} = [-0.0436 \quad -0.9814]$$

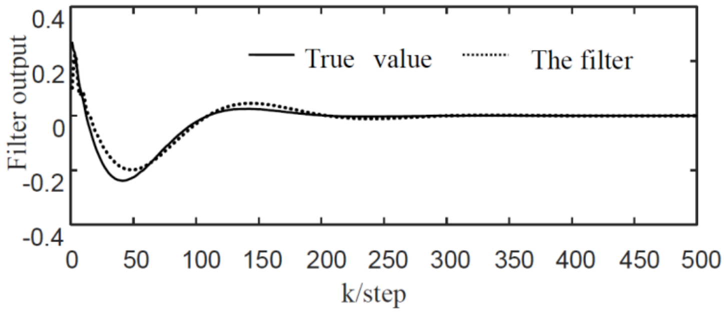

Figure 5.1 The true value of z_k and its H_∞ filter.

Choose the membership functions as $h_1 = \frac{1}{1+e^{-2x_{1k}}}$, $h_2 = 1 - h_1$. The sensor saturations are given as $g(C_{1i}x_k) = \frac{M_1+M_2}{2}C_{1i}x_k + \frac{M_2-M_1}{2}\sin(C_{1i}x_k)$, $\phi(C_{2i}x_k) = \frac{\bar{M}_1+\bar{M}_2}{2}C_{2i}x_k + \frac{\bar{M}_2-\bar{M}_1}{2}\sin(C_{2i}x_k)$ $(i = 1,2)$, where $M_1 = 0.3$, $M_2 = 0.4$, $\bar{M}_1 = 0.6$, $\bar{M}_2 = 0.8$. The external disturbances are $w_k = 1.2sink * e^{-0.2k}$ and $v_k = 1.2sink * e^{-0.2k}$, respectively. The initial values are $x_0 = [0.1 \quad 0.1]^T$ and $\hat{x}_0 = [-0.25 \quad -0.2]^T$. The simulation results are given in Figure 5.1, which shows that the filter we proposed in this chapter is effective.

In order to illustrate the filter designed based on fuzzy-rule-dependent can reduce the conservatism, we compare with the method of fuzzy-rule-independent (that is, the Lyapunov matrix is selected independent of fuzzy rules). The probabilities of $\bar{\xi}_1 = 0.2$ and $\bar{\delta} = 0.6$ are fixed, and $\bar{\xi}_0$ is changed from 0.1 to 0.9, the obtained optimal interference suppression performance γ^* are shown in Table 5.2. From Table 5.2, we can see that the values of γ^* from the fuzzy-rule-dependent method are less than the fuzzy-rule-independent method, which demonstrates that the method of fuzzy-rule-dependent can effectively reduce the conservativeness. At the same time, the H_∞ performance index γ^* is decreasing and the anti-jamming ability of the system is enhancing with the increase of the probability of data receiving on time (*i.e.*, the values of $\bar{\xi}_0$ increase).

Table 5.2 Comparison of γ^* between fuzzy-rule-dependent and fuzzy-rule-independent methods.

$\bar{\xi}_0$ / γ^*	0.1	0.3	0.5	0.7	0.9
Fuzzy-rule-depended method	5.8237	5.7774	5.1049	4.1155	3.2347
Fuzzy-rule-independent method	7.3471	7.2734	6.0407	4.6130	3.4826

Furthermore, we analyze the influences of the main sensor and the redundant sensor on the system performance. When $\bar{\xi}_0$ and $\bar{\xi}_1$ are unchanged, and $\bar{\delta}$ is changed from 0.2 to 1, the results of optimal H_∞ performance index γ^* are given in Table 5.3. We can see that with the increasing of $\bar{\delta}$, the values of γ^* also increase. That is to say, the performance of the standby sensor is better than that of the main sensor.

Table 5.3 The relation of optimal H_∞ performance and the sensor saturation occurrence rate $\bar{\delta}$.

$\bar{\delta}$	0.2	0.4	0.6	0.8	1
γ^*	5.6159	5.7171	5.7774	5.8047	5.8116

In order to illustrate the effects of delay rates and packet loss rates on the system performance, we give the simulation results when the maximum delay is one step and two steps, respectively, which is shown in Table 5.4. There are two cases included in Table 5.4. One is that the on-time rates of a packet are assumed to be the same. If the packet loss rate is also the same, the optimal H_∞ performance index γ^* when the maximum delay is one step is smaller than that when the maximum delay is two steps. That is to say, the larger the transmission delay is, the worse the system performance becomes. The other one is when the probability of the packet received on time increases, as well as the probability of packet loss decreases, γ^* is also reduced and the system performance of interference suppression improves.

Table 5.4 Comparison of γ^* between one-step delay and two-steps delays.

$\bar{\xi}_0/\bar{\xi}_1/\bar{\xi}_2$	$P_{one-delay}$	$P_{two-delay}$	$P_{dropout}$	γ^*
0.2/0.3	0.24	0	0.56	5.8221
0.2/0.1/0.2222	0.08	0.16		16.6811
0.4/0.3	0.21	0	0.42	5.7386
0.4/0.1/0.2222	0.07	0.14		13.6758
0.6/0.3	0.12	0	0.28	4.8987
0.6/0.1/0.2222	0.04	0.08		10.2688
0.8/0.3	0.06	0	0.14	3.7947
0.8/0.1/0.2222	0.02	0.04		8.2085

Finally, we give the simulation results of comparison with reference [22]. The fuzzy H_∞ filter is designed for NCS with sensor saturations in [22], but the designed filter is only depend on one data received at the receiver side. For comparison purpose, let $\bar{\delta} = 0$, the maximum delay of reference [22] is $d = 1$ (*i.e.*, the data is received on time, or received with one-step delay or be lost). On the premise of the same on-time rates and the one-step delay rates, we compare the optimal H_∞ performance index γ^* in Table 5.5. From Table 5.5, we can see that the obtained γ^* by the algorithm we proposed in this chapter are smaller than those in reference [22]. There are two main reasons: one is the multi-packet compensation strategy adopted in our method, the other is the fuzzy-rule-dependent Lyapunov functional we used in the design.

Table 5.5 Comparison of optimal H_∞ performance index γ^* with reference [22].

$\bar{\xi}_0/\bar{\xi}_1$ / γ^*	0.2/0.4	0.3/0.4	0.6/0.7	0.7/0.9
This chapter	5.8443	5.8356	5.8143	5.8023
Reference [22]	7.3637	7.3509	6.5984	5.9142

5.6 Conclusions

The controlled object in networked control system is usually nonlinear, and the sensor for collecting data will also suffer from saturated nonlinear phenomenon. In this chapter, a redundancy strategy is proposed to solve the problem of signal source for the T-S fuzzy system with sensor saturation, and the multi-packet compensation strategy is used to improve the impact of packet loss on the system. Based on the fuzzy-rule-dependent method, a sufficient condition is derived to make the filtering error system asymptotically stable in the mean square sense and satisfy the specified H_∞ performance by using stochastic analysis method. The solution of filter parameters is obtained by using LMI technology. The simulation results show that the method of fuzzy-rule-dependent can achieve less conservativeness than the method based on fuzzy-rule-independent. At the same time, the performance of the system becomes worse with the increase of transmission delay. The T-S model we used in this chapter can approximate the nonlinearity of the system very well, but in practice, the uncertainty of the system is also widespread. There are two ways to deal with the uncertainty of the system. One is to adopt interval type-2 T-S fuzzy model, which uses the upper and lower bounds of membership function to describe the uncertainty of the system. The other is to design the adaptive fuzzy system, *i.e.*, the parameters of the system will be adjusted online by some appropriate methods. These two questions will be the main direction of the authors' future study.

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Chapter 6 H_∞ Control for Networked Systems with Random Delays and Packet Dropouts

6.1 Introduction

In Networked Control Systems (NCSs), the packet dropouts and time delays are the unavoidable phenomena during the information transmission which happen in nature stochastically [1]. Nevertheless, compared to the conventional point-to-point system connection, the NCSs have many advantages, such as easy installation, reduced setup and maintenance costs, *etc*. With the increasing applications in automobiles, manufacturing plants, aircrafts and unmanned vehicles, the NCSs have gained great attention in recent years [2]. Due to the complexity induced by the network, it is a challenging task to analyze and design for such a system [3].

A large number of results focus on the issues of random packet dropouts and random time delays where the associated systems can be described by a stochastic parameter system, see, e.g., references [4-21]. For example, the Kalman filtering and H_2/H_∞ filtering for NCSs with multiple packet dropouts have been studied in [4,5] and [6,7], respectively. Robust filtering has been considered in [8,9] for systems with random delays. Recent literatures [10-12] investigate the filtering problem for systems with random packet dropouts and transmission delays simultaneously. Furthermore, the multi-sensor fusion filtering problems are also discussed in [13-15].

As for the control problem, Sinopoli *et al*. study the LQG controller for systems with packet dropouts under the TCP and UDP types of networking protocols in [16,17]. The H_∞ controllers via an LMI approach for systems with random packet dropouts are considered in [18] and [19]. Yang *et al*. investigate the H_∞ control for networked systems with random time delays in [20]. The latest references [21,22] consider the control problem for systems with random packet dropouts and time delays simultaneously. [21] gives an H_∞ control based on the observer, the controller and observer gains can be obtained by solving the LMIs. But the model established can only described the phenomenon of random consecutive packet dropouts or the time delays. Thus, it cannot deal with the random packet dropouts and time delays simultaneously.

[22] gives a new switched model to describe the packet dropouts and time delays, and the sufficient condition is obtained which establishes the quantitative relation between the packet-dropout rate and the stability of the NCS. But the delay is assumed to be constant.

So far, the H_∞ control synthesis for the NCSs with both random time delays and packet dropouts has not been fully investigated. Motivated by the above-mentioned problem, we study the H_∞ control in this chapter for a class of NCSs with random packet dropouts and time delays which happen in both the sensor-to-estimator channel and the controller-to-actuator channel. A novel model is established by using four Bernoulli distributed stochastic variables to describe the phenomena of the random multiple packet dropouts and one-step time delays. An observer-based controller is designed via an LMI approach such that the closed-loop networked control system is exponentially stable in the sense of mean square, and the prescribed disturbance attenuation performance is achieved. The simulation results on an uninterruptible power system (UPS) demonstrate the effectiveness of the proposed method. The main contributions of this chapter different from the previous work [21] are as follows. (i) A novel model is presented to describe the random multiple packet dropouts and one-step time delay simultaneously where two stochastic variables satisfying Bounulli distributions are used in one channel. (ii) The cone complementarity linearization method [23] is used to obtain the controller gain. Differently from [23], we add the disturbance-rejection-attenuation level γ into the minimizing performance index such that the H_∞ performance of the closed-loop system can be obtained simultaneously.

6.2 Problem formulation

Consider the discrete-time linear stochastic system:

$$\begin{cases} x_{k+1} = Ax_k + B_2u_k + B_1w_k \\ \tilde{y}_k = C_2x_k + C_1w_k \\ z_k = D_2x_k + D_1w_k \end{cases} \tag{6.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $\tilde{y}_k \in \mathbb{R}^r$ is the measured output, $z_k \in \mathbb{R}^m$ is the controlled output, $w_k \in \mathbb{R}^q$ is the disturbance input belonging to $l_2[0,\infty)$, $u_k \in \mathbb{R}^p$ is the controller received by the actuator, and A, B_1, B_2, C_1, C_2, D_1 and D_2 are known constant matrices with appropriate dimensions.

Because of the unreliable transmission media, there will be random packet dropouts and time delays in the sensor-to-estimator and controller-to-actuator network links. To avoid the network congestion, we assume that a packet is only transmitted one time. We first consider the sensor-to-estimator channel, the measurement received by the estimator via an unreliable data communication is described by:

$$y_k = \xi_k \tilde{y}_k + (1-\xi_k)(1-\xi_{k-1})\delta_k \tilde{y}_{k-1} + (1-\xi_k)[1-(1-\xi_{k-1})\delta_k]y_{k-1} \tag{6.2}$$

where $y_k \in \mathbb{R}^r$, ξ_k and δ_k are the uncorrelated Bernoulli distributed random variables that satisfy probabilities $\text{Prob}\{\xi_k = 1\} = \bar{\xi}$, $\text{Prob}\{\xi_k = 0\} = 1 - \bar{\xi}$ and $\text{Prob}\{\delta_k = 1\} = \bar{\delta}$, $\text{Prob}\{\delta_k = 0\} = 1 - \bar{\delta}$, where $0 \le \bar{\xi}, \bar{\delta} \le 1$.

Remark 6.1 For the model (6.2), if $\xi_k = 0$ and $\xi_{k-1} = 1$ or $\delta_k = 0$, it is reduced to the packet dropouts case (the estimator does not receive the measured output at time k), then $y_k = y_{k-1}$, *i.e.*, the received measurement at $k-1$ is used at k. If $\xi_k = 0$, $\xi_{k-1} = 0$ and $\delta_k = 1$, it is reduced to the case of possible one-unit delay (the estimator receives a measured output of $\mathrm{k}-1$ at k). If $\xi_k = 1$, it is reduced to the case of no delays and packet dropouts (the estimator receives the measured output on time).

Remark 6.2 It is not difficult to see that the on-time rate for a packet of the sensor at instant k to be received by the estimator is $\text{Prob}\{\xi_k = 1\} = \bar{\xi}$, one-step delay rate is $\text{Prob}\{\xi_k = 0, \xi_{k-1} = 0, \delta_k = 1\} = (1-\bar{\xi})^2\bar{\delta}$, and packet dropout rate is

$$\text{Prob}\{\xi_k = 0, \xi_{k-1} = 1\} + \text{Prob}\{\xi_k = 0, \xi_{k-1} = 0, \delta_k = 0\} = \left(1-\bar{\xi}\right)\bar{\xi} + \left(1-\bar{\xi}\right)^2\left(1-\bar{\delta}\right).$$

Remark 6.3 It is clear that the model (6.2) is more general than the model proposed in [4,21] where the multiple packet dropouts are only described. When $\delta_k = 0$, it is reduced to the model described in [4,21].

Similar to the measurement channel, the control signals sent by the remote controller to the actuator via the communication channel can be described by

$$u_k = \alpha_k \tilde{u}_k + (1-\alpha_k)(1-\alpha_{k-1})\beta_k \tilde{u}_{k-1} + (1-\alpha_k)[1-(1-\alpha_{k-1})\beta_k]u_{k-1} \tag{6.3}$$

where $\tilde{u}_k \in \mathbb{R}^p$ is the control input to be designed. α_k and β_k are the uncorrelated Bernoulli distributed random variables that satisfy probabilities $\text{Prob}\{\alpha_k = 1\} = \bar{\alpha}$, $\text{Prob}\{\alpha_k = 0\} = 1 - \bar{\alpha}$ and $\text{Prob}\{\beta_k = 1\} = \bar{\beta}$, $\text{Prob}\{\beta_k = 0\} = 1 - \bar{\beta}$, where $0 \le \bar{\alpha}, \bar{\beta} \le 1$. The analysis of the probability distribution is similar with the model (6.2).

To obtain the compact form by augmentation, we will give the following definitions:

$$\tilde{Y}_k = (1-\xi_k)\tilde{y}_k,\ Y_k = \xi_k y_k,\ \tilde{U}_k = (1-\alpha_k)\tilde{u}_k,\ U_k = \alpha_k u_k.$$

From (6.1)-(6.3), we have the NCSs formulation with random multiple packet dropouts and one-step time delays which can be augmented as follows:

$$\begin{cases} X_{k+1} = \widetilde{\Phi}_k X_k + \tilde{B}_k \tilde{u}_k + \tilde{\Gamma}_k w_k \\ y_k = \widetilde{H}_k X_k + \xi_k C_1 w_k \\ z_k = \overline{D}_2 X_k + D_1 w_k \end{cases} \tag{6.4}$$

where

$$X_{k+1} = [x_{k+1}^T \quad \tilde{Y}_k^T \quad Y_k^T \quad y_k^T \quad \widetilde{U}_k^T \quad U_k^T \quad u_k^T]^T,$$

$$\widetilde{\Phi}_k = \Phi_0 + \xi_k \Phi_1 + (1-\xi_k)\delta_k \Phi_2 + \alpha_k \Phi_3 + (1-\alpha_k)\beta_k \Phi_4,$$

$$\widetilde{H}_k = H_0 + \xi_k H_1 + (1-\xi_k)\delta_k H_2,$$

$$\tilde{B}_k = B_{20} + \alpha_k B_{21},\ \tilde{\Gamma}_k = \Gamma_0 + \xi_k \Gamma_1,$$

and $\Phi_i (i = 0,1,\cdots,4)$; B_{20}, B_{21}, Γ_0, Γ_1, $H_i (i = 0,1,2)$ and $\overline{D}_2$ are defined by

$$\Phi_0 = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & B_2 \\ C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},\ \Phi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_2 & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Phi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & I & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ \Phi_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix}, \tag{6.5}$$

$$\Phi_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & B_2 & B_2 & -B_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & -I & -I \end{bmatrix},\ B_{20} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix},\ B_{21} = \begin{bmatrix} B_2 \\ 0 \\ 0 \\ 0 \\ -I \\ I \\ I \end{bmatrix},\ \Gamma_0 = \begin{bmatrix} B_1 \\ C_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \ \Gamma_1 = \begin{bmatrix} 0 \\ -C_1 \\ C_1 \\ C_1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$H_0 = [0 \quad 0 \quad 0 \quad I \quad 0 \quad 0 \quad 0],\ H_1 = [C_2 \quad 0 \quad 0 \quad -I \quad 0 \quad 0 \quad 0],$$

$$H_2 = [0 \quad I \quad I \quad -I \quad 0 \quad 0 \quad 0],\ \overline{D}_2 = [D_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

Remark 6.4 From the distributions of $\xi_k, \delta_k, \alpha_k, \beta_k$ which are uncorrelated with each other, we can easily to obtain that

$$E\{\xi_k(1-\xi_k)\} = 0,\ E\{\alpha_k(1-\alpha_k)\} = 0,$$

$$E\{(\xi_k - \bar{\xi})^2\} = \bar{\xi}(1-\bar{\xi}),\ E\{(\alpha_k - \bar{\alpha})^2\} = \bar{\alpha}(1-\bar{\alpha}),$$

$$E\{[(1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta}]^2\} = (1-\bar{\xi})\bar{\delta}[1-(1-\bar{\xi})\bar{\delta}],$$

$$E\{[(1-\alpha_k)\beta_k - (1-\bar{\alpha})\bar{\beta}]^2\} = (1-\bar{\alpha})\bar{\beta}[1-(1-\bar{\alpha})\bar{\beta}].$$

In this chapter, we propose a dynamic observer-based control scheme for the system (6.4) described by

$$\begin{cases} \hat{X}_{k+1} = \overline{\Phi}\hat{X}_k + \overline{B}\tilde{u}_k + L(y_k - \hat{y}_k) \\ \hat{y}_k = \overline{H}\hat{X}_k \\ \tilde{u}_k = K\hat{X}_k \end{cases} \tag{6.6}$$

where $\overline{\Phi} = E\{\widetilde{\Phi}_k\}$, $\overline{B} = E\{\tilde{B}_k\}$, $\overline{\Gamma} = E\{\tilde{\Gamma}_k\}$, $\overline{H} = E\{\widetilde{H}_k\}$, which are given in (6.5) where ξ_k, δ_k, α_k and β_k are replaced by their expectations $\bar{\xi}$, $\bar{\delta}$, $\bar{\alpha}$ and $\bar{\beta}$. K and L

are the control gain and observer gain, respectively.

Let the estimation error be

$$e_k = X_k - \hat{X}_k \tag{6.7}$$

The closed-loop system can be obtained by substituting (6.2), (6.3) and (6.6) into (6.4) and (6.7)

$$\begin{aligned} X_{k+1} &= [(\tilde{\Phi}_k - \bar{\Phi}) + (\tilde{B}_k - \bar{B})K]X_k + (\bar{\Phi} + \bar{B}K)X_k - (\tilde{B}_k - \bar{B})Ke_k - \bar{B}Ke_k + (\tilde{\Gamma}_k - \bar{\Gamma})w_k \\ &\quad + \bar{\Gamma}w_k \\ e_{k+1} &= [(\tilde{\Phi}_k - \bar{\Phi}) + (\tilde{B}_k - \bar{B})K - L(\tilde{H}_k - \bar{H})]X_k + (\bar{\Phi} - L\bar{H})e_k - (\tilde{B}_k - \bar{B})Ke_k \\ &\quad + [(\tilde{\Gamma}_k - \bar{\Gamma}) - LC_1(\xi_k - \bar{\xi})]w_k + (\bar{\Gamma} - LC_1\bar{\xi})w_k \end{aligned} \tag{6.8}$$

or in an augmented form as:

$$\begin{aligned} \begin{bmatrix} X_{k+1} \\ e_{k+1} \end{bmatrix} &= \begin{bmatrix} \bar{\Phi} + \bar{B}K & -\bar{B}K \\ 0 & \bar{\Phi} - L\bar{H} \end{bmatrix} \begin{bmatrix} X_k \\ e_k \end{bmatrix} + \\ &\begin{bmatrix} (\tilde{\Phi}_k - \bar{\Phi}) + (\tilde{B}_k - \bar{B})K & -(\tilde{B}_k - \bar{B})K \\ (\tilde{\Phi}_k - \bar{\Phi}) + (\tilde{B}_k - \bar{B})K - L(\tilde{H}_k - \bar{H}) & -(\tilde{B}_k - \bar{B})K \end{bmatrix} \begin{bmatrix} X_k \\ e_k \end{bmatrix} + \begin{bmatrix} \bar{\Gamma} \\ \bar{\Gamma} - LC_1\bar{\xi} \end{bmatrix} w_k + \\ &\begin{bmatrix} \tilde{\Gamma}_k - \bar{\Gamma} \\ (\tilde{\Gamma}_k - \bar{\Gamma}) - LC_1(\xi_k - \bar{\xi}) \end{bmatrix} w_k \end{aligned} \tag{6.9}$$

Let $\eta_k = [X_k^T \quad e_k^T]^T$, we can rewrite (6.9) into a compact form as follows:

$$\begin{aligned} \eta_{k+1} &= \big[\breve{A} + (\xi_k - \bar{\xi})\breve{A}_1 + (1 - \xi_k)\delta_k - (1 - \bar{\xi})\bar{\delta})\breve{A}_2 + (\alpha_k - \bar{\alpha})\breve{A}_3 + ((1 - \alpha_k)\beta_k - \\ &\quad (1 - \bar{\alpha})\bar{\beta})\breve{A}_4]\eta_k + (\breve{B} + (\xi_k - \bar{\xi})\breve{B}_1)w_k \end{aligned} \tag{6.10}$$

where

$$\breve{A} = \begin{bmatrix} \bar{\Phi} + \bar{B}K & -\bar{B}K \\ 0 & \bar{\Phi} - L\bar{H} \end{bmatrix}, \breve{A}_1 = \begin{bmatrix} \Phi_1 & 0 \\ \Phi_1 - LH_1 & 0 \end{bmatrix}, \breve{A}_2 = \begin{bmatrix} \Phi_2 & 0 \\ \Phi_2 - LH_2 & 0 \end{bmatrix}$$

$$\breve{A}_3 = \begin{bmatrix} \Phi_3 + B_{21}K & -B_{21}K \\ \Phi_3 + B_{21}K & -B_{21}K \end{bmatrix}, \ \breve{A}_4 = \begin{bmatrix} \Phi_4 & 0 \\ \Phi_4 & 0 \end{bmatrix}, \ \breve{B} = \begin{bmatrix} \bar{\Gamma} \\ \bar{\Gamma} - LC_1\bar{\xi} \end{bmatrix}, \ \breve{B}_1 = \begin{bmatrix} \Gamma_1 \\ \Gamma_1 - LC_1 \end{bmatrix}$$

Definition 6.1 The closed-loop system (6.10) is said to be exponentially mean-square stable if when $w_k = 0$, there exist constants $\varphi > 0$ and $\tau \in (0,1)$, such that

$$E\{\|\eta_k\|^2\} \leq \varphi\tau^k E\{\|\eta_0\|^2\}, \text{ for all } \eta_0 \in \mathbb{R}^{2n}, k \in I^+.$$

Definition 6.2 The closed-loop system (6.10) is said to be exponential mean-square stability with an H_∞ disturbance attenuation level γ, if system (6.10) is mean-square stable and for the zero-initial condition, the controlled output z_k satisfies

$$\textstyle\sum_{k=0}^{\infty} E\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\|w_k\|^2\}, \tag{6.11}$$

where $\gamma > 0$ is a prescribed scalar.

With the Definitions 6.1 and 6.2, our aim is to design a controller such that the closed-loop system (6.10) satisfies the following two performance requirements (Q1) and (Q2).

(Q1) The closed-loop system (6.10) is exponentially mean-square stable.

(Q2) The closed-loop system (6.10) satisfies the H_∞ performance constraint.

6.3 Stability analysis

In this section, we establish a sufficient condition of the exponentially mean-square stability for the given controller and observer gains K and L of the closed-loop system (6.10), which will be the fundamental of the H_∞ controller design in the next section.

Lemma 6.1 [24] Let $\mathrm{V}(\eta_k)$ be a Lyapunov functional. If there exist real scalars $\lambda \geq 0,\ \mu > 0,\ \nu > 0$, and $0 < \psi < 1$ such that

$$\mu \parallel \eta_k \parallel^2 \leq \mathrm{V}(\eta_k) \leq \nu \parallel \eta_k \parallel^2$$

$$E\{\mathrm{V}(\eta_{k+1}|\eta_k)\} - \mathrm{V}(\eta_k) \leq \lambda - \psi \mathrm{V}(\eta_k)$$

then the sequence η_k satisfies

$$E\{\parallel \eta_k \parallel^2\} \leq \frac{\nu}{\mu} \parallel \eta_0 \parallel^2 (1-\psi)^k + \frac{\lambda}{\mu\psi}$$

Theorem 6.1 Suppose that both the controller gain matrix K and the observer gain matrix L are given. The closed-loop system (6.10) is exponentially mean-square stable if there exist positive definite matrices P and Q satisfying

$$\begin{bmatrix} -P & * & * & * \\ 0 & -Q & * & * \\ \Xi_{11} & \Xi_{21} & -\Xi_{31} & * \\ \Xi_{12} & \Xi_{22} & 0 & -\Xi_{32} \end{bmatrix} < 0 \tag{6.12}$$

where

$$\Xi_{11} = [(\bar{\Phi} + \bar{B}K)^T \quad 0 \quad \Phi_1^T \quad (\Phi_1 - LH_1)^T \quad \Phi_2^T \quad (\Phi_2 - LH_2)^T \quad (\Phi_1 - \Phi_2)^T \quad (\Phi_1 - LH_1 - \Phi_2 + LH_2)^T]^T$$

$$\Xi_{12} = [(\Phi_3 + B_{21}K)^T \quad (\Phi_3 + B_{21}K)^T \quad \Phi_4^T \quad \Phi_4^T \quad (\Phi_3 + B_{21}K - \Phi_4)^T \quad (\Phi_3 + B_{21}K - \Phi_4)^T]^T$$

$$\Xi_{21} = [-(\bar{B}K)^T \quad (\bar{\Phi} - L\bar{H})^T \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

$$\Xi_{22} = [-(B_{21}K)^T \quad -(B_{21}K)^T \quad 0 \quad 0 \quad -(B_{21}K)^T \quad -(B_{21}K)^T]^T$$

$$\Xi_{31} = diag\{P^{-1} \quad Q^{-1} \quad \theta_1^{-2}P^{-1} \quad \theta_1^{-2}Q^{-1} \quad \theta_2^{-2}P^{-1} \quad \theta_2^{-2}Q^{-1} \quad \theta_3^{-2}P^{-1} \quad \theta_3^{-2}Q^{-1}\}$$

$$\Xi_{32} = diag\{\theta_4^{-2}P^{-1} \quad \theta_4^{-2}Q^{-1} \quad \theta_5^{-2}P^{-1} \quad \theta_5^{-2}Q^{-1} \quad \theta_6^{-2}P^{-1} \quad \theta_6^{-2}Q^{-1}\}$$

$$\theta_1^2 = \bar{\xi}(1-\bar{\xi})(1-\bar{\delta}),\ \theta_2^2 = (1-\bar{\xi})^2(1-\bar{\delta})\bar{\delta},$$

$$\theta_3^2 = \bar{\alpha}(1-\bar{\alpha})(1-\bar{\beta}),\ \theta_4^2 = (1-\bar{\alpha})^2(1-\bar{\beta})\bar{\beta},$$

$$\theta_5^2 = \bar{\xi}(1-\bar{\xi})\bar{\delta},\ \theta_6^2 = \bar{\alpha}(1-\bar{\alpha})\bar{\beta}.$$

Proof We now consider the exponential mean-square stability of system (6.10) with $w_k = 0$.

Define a Lyapunov function

$$V_k = X_k^T P X_k + e_k^T P e_k \tag{6.13}$$

which can be written as a compact form of $V_k = \eta_k^T \tilde{P} \eta_k$ by denoting $\tilde{P} = diag(P, Q)$. Then we have

$$\Delta V_k = E\{V_{k+1}|X_k,\cdots,X_0,e_k,\cdots,e_0\} - V_k = E\{\eta_{k+1}^T\tilde{P}\eta_{k+1}\} - \eta_k^T\tilde{P}\eta_k$$
$$\begin{aligned} &= E\Big\{\{\eta_k^T[\breve{A} + (\xi_k - \bar{\xi})\breve{A}_1 + (1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta})\breve{A}_2 + (\alpha_k - \bar{\alpha})\breve{A}_3 + ((1-\alpha_k)\beta_k \\ &\quad - (1-\bar{\alpha})\bar{\beta})\breve{A}_4]^T\tilde{P}[\breve{A} + (\xi_k - \bar{\xi})\breve{A}_1 + (1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta})\breve{A}_2 \\ &\quad + (\alpha_k - \bar{\alpha})\breve{A}_3 + \big((1-\alpha_k)\beta_k - (1-\bar{\alpha})\bar{\beta}\big)\breve{A}_4\Big] - \eta_k^T\tilde{P}\eta_k \end{aligned} \tag{6.14}$$

From Remark 6.4, we have

$$\begin{aligned} \Delta V_k &= \eta_k^T\Big[-\tilde{P} + \breve{A}^T\tilde{P}\breve{A} + E\left\{(\xi_k - \bar{\xi})^2\right\}\breve{A}_1^T\tilde{P}\breve{A}_1 + E\left\{\left[(1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta}\right]^2\right\}\breve{A}_2^T\tilde{P}\breve{A}_2 \\ &\quad + E\{(\alpha_k - \bar{\alpha})^2\}\breve{A}_3^T\tilde{P}\breve{A}_3 + E\left\{\left[(1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta}\right]^2\right\}\breve{A}_4^T\tilde{P}\breve{A}_4 \\ &\quad + E\{(\xi_k - \bar{\xi})[(1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta}]\}(\breve{A}_1^T\tilde{P}\breve{A}_2 + \breve{A}_2^T\tilde{P}\breve{A}_1) \\ &\quad + E\{(\alpha_k - \bar{\alpha})[(1-\alpha_k)\beta_k - (1-\bar{\alpha})\bar{\beta}]\}(\breve{A}_3^T\tilde{P}\breve{A}_4 + \breve{A}_4^T\tilde{P}\breve{A}_3)\Big]\eta_k \\ &= \eta_k^T\Big[-\tilde{P} + \breve{A}^T\tilde{P}\breve{A} + \theta_1^2\breve{A}_1^T\tilde{P}\breve{A}_1 + \theta_2^2\breve{A}_2^T\tilde{P}\breve{A}_2 + \theta_3^2\breve{A}_3^T\tilde{P}\breve{A}_3 + \theta_4^2\breve{A}_4^T\tilde{P}\breve{A}_4 + \theta_5^2(\breve{A}_1 - \breve{A}_2)^T\tilde{P}(\breve{A}_1 - \\ &\quad \breve{A}_2) + \theta_6^2(\breve{A}_3 - \breve{A}_4)^T\tilde{P}(\breve{A}_3 - \breve{A}_4)\Big]\eta_k = \eta_k^T\Omega\eta_k \end{aligned} \tag{6.15}$$

where

$$\theta_5^2 = -E\{(\xi_k - \bar{\xi})[(1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta}]\} = \bar{\xi}(1-\bar{\xi})\bar{\delta}$$
$$\theta_1^2 = E\left\{(\xi_k - \bar{\xi})^2\right\} - \theta_5^2 = \bar{\xi}(1-\bar{\xi})(1-\bar{\delta})$$
$$\theta_2^2 = E\left\{\left[(1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta}\right]^2\right\} - \theta_5^2 = (1-\bar{\xi})^2(1-\bar{\delta})\bar{\delta},$$
$$\theta_6^2 = -E\{(\alpha_k - \bar{\alpha})[(1-\alpha_k)\beta_k - (1-\bar{\alpha})\bar{\beta}]\} = \bar{\alpha}(1-\bar{\alpha})\bar{\beta}$$
$$\theta_3^2 = E\{(\alpha_k - \bar{\alpha})^2\} - \theta_6^2 = \bar{\alpha}(1-\bar{\alpha})(1-\bar{\beta})$$
$$\theta_4^2 = E\{[(1-\alpha_k)\beta_k - (1-\bar{\alpha})\bar{\beta}]^2\} - \theta_6^2 = (1-\bar{\alpha})^2(1-\bar{\beta})\bar{\beta}$$

According to the definitions of $\breve{A}$, $\breve{A}_i (i = 1,\cdots,4)$ and $\tilde{P}$, Ω can be given as follows:

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_2^T & \Omega_3 \end{bmatrix} \tag{6.16}$$

with

$$\begin{aligned} \Omega_1 &= (\bar{\Phi} + \bar{B}K)^TP(\bar{\Phi} + \bar{B}K) + \theta_1^2\Phi_1^TP\Phi_1 + \theta_1^2(\Phi_1 - LH_1)^TQ(\Phi_1 - LH_1) + \theta_2^2\Phi_2^TP\Phi_2 \\ &\quad + \theta_2^2(\Phi_2 - LH_2)^TQ(\Phi_2 - LH_2) + \theta_3^2(\Phi_3 + B_{21}K)^TP(\Phi_3 + B_{21}K) \\ &\quad + \theta_3^2(\Phi_3 + B_{21}K)^TQ(\Phi_3 + B_{21}K) + \theta_4^2\Phi_4^TP\Phi_4 + \theta_4^2\Phi_4^TQ\Phi_4 \\ &\quad + \theta_5^2(\Phi_1 - \Phi_2)^TP(\Phi_1 - \Phi_2) \\ &\quad + \theta_5^2(\Phi_1 - LH_1 - \Phi_2 + LH_2)^TQ(\Phi_1 - LH_1 - \Phi_2 + LH_2) \\ &\quad + \theta_6^2(\Phi_3 + B_{21}K - \Phi_4)^TP(\Phi_3 + B_{21}K - \Phi_4) \\ &\quad + \theta_6^2(\Phi_3 + B_{21}K - \Phi_4)^TQ(\Phi_3 + B_{21}K - \Phi_4) - P \\ \Omega_2 &= -(\bar{\Phi} + \bar{B}K)^TP(\bar{B}K) - \theta_3^2(\Phi_3 + B_{21}K)^TP(B_{21}K) - \theta_3^2(\Phi_3 + B_{21}K)^TQ(B_{21}K) \\ &\quad - \theta_6^2(\Phi_3 + B_{21}K - \Phi_4)^TQ(B_{21}K) \\ \Omega_3 &= -Q + (\bar{B}K)^TP(\bar{B}K) + (\bar{\Phi} - L\bar{H})^TQ(\bar{\Phi} - L\bar{H}) + \theta_3^2(B_{21}K)^TP(B_{21}K) \\ &\quad + \theta_3^2(B_{21}K)^TQ(B_{21}K) \end{aligned} \tag{6.17}$$

By using the Shur complement, (6.12) implies $\Omega < 0$. Thus, we have

$$\Delta V_k = E\{V_{k+1}|X_k,\cdots,X_0,e_k,\cdots,e_0\} - V_k = \eta_k^T\Omega\eta_k \leq -\lambda_{min}(-\Omega)\eta_k^T\eta_k < -\phi\eta_k^T\eta_k \tag{6.18}$$

where

$$0 < \phi < \min\{-\lambda_{min}(-\Omega),\kappa\},\ \ \kappa = \max\{\lambda_{max}(\mathrm{P}),\lambda_{max}(\mathrm{Q})\}.$$

Form (6.18), we have

$$\Delta V_k = E\{V_{k+1}|X_k,\cdots,X_0,e_k,\cdots,e_0\} - V_k < -\frac{\phi}{\kappa}V_k = -\psi V_k$$

where $\psi = \frac{\phi}{\kappa}$. By Definition 6.1 and Lemma 6.1, it can be verified that the closed-loop system (6.10) is exponentially mean-square stable.

6.4 H_∞ controller design

In this section, we give a design approach of H_∞ controller based on the result obtained in the previous section with the consideration of random packet dropouts and time delays, simultaneously.

Theorem 6.2 Suppose that both the controller gain matrix K and the observer gain matrix L are given. The system (6.10) is exponentially mean-square stable and the H_∞ norm constraint (6.11) is achieved for a given scalar $\gamma > 0$ when $w_k \neq 0$ if there exist positive definite matrices P and Q satisfying

$$\begin{bmatrix} -P & * & * & * & * \\ 0 & -Q & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ \bar{\Xi}_{11} & \bar{\Xi}_{21} & \bar{\Xi}_{31} & -\bar{\Xi}_{41} & * \\ \bar{\Xi}_{12} & \bar{\Xi}_{22} & 0 & D_1 & -\bar{\Xi}_{42} \end{bmatrix} < 0 \tag{6.19}$$

where

$$\bar{\Xi}_{11} = [(\bar{\Phi}+\bar{B}K)^T\ 0\ \theta_1\Phi_1^T\ \theta_1(\Phi_1-LH_1)^T\ \theta_2\Phi_2^T\ \theta_2(\Phi_2-LH_2)^T\ \theta_3(\Phi_1-\Phi_2)^T\ \theta_3(\Phi_1-LH_1-\Phi_2+LH_2)^T]^T$$

$$\Xi_{12} = [\theta_4(\Phi_3+B_{21}K)^T\quad \theta_4(\Phi_3+B_{21}K)^T\quad \theta_5\Phi_4^T\quad \theta_5\Phi_4^T\quad \theta_6(\Phi_3+B_{21}K-\Phi_4)^T\quad \theta_6(\Phi_3+B_{21}K-\Phi_4)^T\quad \bar{D}_2^T]^T$$

$$\bar{\Xi}_{21} = [-(\bar{B}K)^T\quad (\bar{\Phi}-L\bar{H})^T\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0]^T$$

$$\bar{\Xi}_{22} = [-\theta_3(B_{21}K)^T\quad -\theta_3(B_{21}K)^T\quad 0\quad 0\quad -\theta_6(B_{21}K)^T\quad -\theta_6(B_{21}K)^T]^T,$$

$$\bar{\Xi}_{31} = \left[-\bar{\Gamma}^T\quad \left(\bar{\Gamma}-LC_1\bar{\xi}\right)^T\quad \theta_1\Gamma_1^T\quad \theta_1(\Gamma_1-LC_1)^T\quad 0\quad 0\quad \theta_3\Gamma_1^T\quad \theta_3(\Gamma_1-LC_1)^T\right]^T,$$

$$\bar{\Xi}_{41} = diag\{P^{-1}\quad Q^{-1}\quad P^{-1}\quad Q^{-1}\quad P^{-1}\quad Q^{-1}\quad P^{-1}\quad Q^{-1}\}$$

$$\bar{\Xi}_{42} = diag\{P^{-1}\quad Q^{-1}\quad P^{-1}\quad Q^{-1}\quad P^{-1}\quad Q^{-1}\quad I\}$$

Proof It is obvious that (6.19) implies (6.12), hence it follows from Theorem 6.1 that the system (6.10) is exponentially mean-square stable. Next, we prove that the closed-loop system is with H_∞ performance level γ for any nonzero w_k. Denote that $\varsigma_k = [\eta_k^T\quad w_k^T]^T$, it follows from (6.10) and (6.15) that

$$\Delta V_k = E\{V_{k+1}|X_k,\cdots,X_0,e_k,\cdots,e_0\} - V_k + E\{z_k^T z_k\} - \gamma^2 E\{w_k^T w_k\}$$

$$\begin{aligned}
&= E\Big\{\{\eta_k^T[\breve{A} + (\xi_k - \bar{\xi})\breve{A}_1 + (1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta})\breve{A}_2 + (\alpha_k - \bar{\alpha})\breve{A}_3 + ((1-\alpha_k)\beta_k \\
&\quad - (1-\bar{\alpha})\bar{\beta})\breve{A}_4]^T \tilde{P}[\breve{A} + (\xi_k - \bar{\xi})\breve{A}_1 + (1-\xi_k)\delta_k - (1-\bar{\xi})\bar{\delta})\breve{A}_2 \\
&\quad + (\alpha_k - \bar{\alpha})\breve{A}_3 + \big((1-\alpha_k)\beta_k - (1-\bar{\alpha})\bar{\beta}\big)\breve{A}_4\big] - \eta_k^T \tilde{P}\eta_k \\
&\quad + E\{(\bar{D}_2 X_k + D_1 w_k)^T(\bar{D}_2 X_k + D_1 w_k) - \gamma^2 w_k^T w_k\} \\
&= \eta_k^T\Big[-\tilde{P} + \breve{A}^T\tilde{P}\breve{A} + \theta_1^2\breve{A}_1^T\tilde{P}\breve{A}_1 + \theta_2^2\breve{A}_2^T\tilde{P}\breve{A}_2 + \theta_3^2\breve{A}_3^T\tilde{P}\breve{A}_3 + \theta_4^2\breve{A}_4^T\tilde{P}\breve{A}_4 \\
&\quad + \theta_5^2(\breve{A}_1 - \breve{A}_2)^T\tilde{P}(\breve{A}_1 - \breve{A}_2) + \theta_6^2(\breve{A}_3 - \breve{A}_4)^T\tilde{P}(\breve{A}_3 - \breve{A}_4)\Big]\eta_k + w_k^T\breve{B}^T\tilde{P}\breve{B}w_k \\
&\quad + (\theta_1^2 + \theta_5^2)w_k^T\breve{B}_1^T\tilde{P}\breve{B}_1 w_k + w_k^T D_1^T D_1 w_k - \gamma^2 w_k^T w_k + X_k^T\bar{D}_2^T\bar{D}_2 X_k \\
&\quad + X_k^T\bar{D}_2^T D_1 w_k + w_k^T D_1^T\bar{D}_2 X_k + \eta_k^T\breve{A}^T\tilde{P}\breve{B}w_k + w_k^T\breve{B}^T\tilde{P}\breve{A}\eta_k \\
&\quad + (\theta_1^2 + \theta_5^2)\eta_k^T\breve{A}_1^T\tilde{P}\breve{B}_1 w_k + (\theta_1^2 + \theta_5^2)w_k^T\breve{B}^T\tilde{P}\breve{A}_1\eta_k - \theta_5^2\eta_k^T\breve{A}_2^T\tilde{P}\breve{B}w_k \\
&\quad - \theta_5^2 w_k^T\breve{B}^T\tilde{P}\breve{A}_2\eta_k = \varsigma_k^T\bar{\Omega}\varsigma_k
\end{aligned} \tag{6.20}$$

where

$$\bar{\Omega} = \begin{bmatrix} \Omega + \Omega_4 & * \\ \Omega_5 & \Omega_6 \end{bmatrix} \tag{6.21}$$

with

$$\begin{aligned}
&\Omega_4 = \begin{bmatrix} \bar{D}_2^T\bar{D}_2 & 0 \\ 0 & 0 \end{bmatrix},\ \Omega_5 = [\Omega_{51} \quad \Omega_{52}], \\
&\Omega_{51} = D_1^T\bar{D}_2 + \bar{\Gamma}^T P(\bar{\Phi} + \bar{B}K) + \theta_1^2\Gamma_1^T P\Phi_1 + \theta_1^2(\Gamma_1 - LC_1)^T Q(\Phi_1 - LH_1) + \theta_5^2\Gamma_1^2 P(\Phi_1 - \Phi_2) \\
&\qquad + \theta_5^2(\Gamma_1 - LC_1)^T Q(\Phi_1 - LH_1 - \Phi_2 - LH_2) \\
&\Omega_{52} = \bar{\Gamma}^T P(-\bar{B}K) + (\bar{\Gamma} - \bar{\xi}LC_1)^T Q(\bar{\Phi} - L\bar{H}) \\
&\Omega_6 = \bar{\Gamma}^T P\bar{\Gamma} + (\bar{\Gamma} - \bar{\xi}LC_1)^T Q(\bar{\Gamma} - \bar{\xi}LC_1) + \theta_1^2\bar{\Gamma}_1^T P\bar{\Gamma}_1 + \theta_1^2(\Gamma_1 - LC_1)^T Q(\Gamma_1 - LC_1) + \theta_5^2\bar{\Gamma}_1^T P\bar{\Gamma}_1 \\
&\qquad + \theta_5^2(\Gamma_1 - LC_1)^T Q(\Gamma_1 - LC_1) + D_1^T D_1 - \gamma^2 I
\end{aligned} \tag{6.22}$$

Obviously, by using the Shur complement, (6.19) implies $\bar{\Omega} < 0$. Thus, we have

$$E\{V_{k+1}|X_k,\cdots,X_0,e_k,\cdots,e_0\} - V_k + E\{z_k^T z_k\} - \gamma^2 E\{w_k^T w_k\} = E\{\varsigma_k^T\bar{\Omega}\varsigma_k\} \tag{6.23}$$

Summing up (6.23) from 0 to ∞ with respect to k yields

$$E\{z_k^T z_k\} < \gamma^2 E\{w_k^T w_k\} - E\{V_\infty\} + E\{V_0\}.$$

Since the closed-loop system (6.10) is exponentially mean-square stable and by using zero initial condition, the following is obtained immediately.

$$\sum_{k=0}^{\infty} E\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\|w_k\|^2\}$$

This completes the proof.

Theorem 6.3 Given a scalar $\gamma > 0$. The system (6.10) is exponentially mean-square stable and the H_∞ norm constraint (6.11) is achieved for all nonzero w_k if there exist positive definite matrices P, Q, S, and R satisfying $PS = I$, $QR = I$ and real matrices K and L such that

$$\begin{bmatrix} -P & * & * & * & * \\ 0 & -Q & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * \\ \bar{\bar{\Xi}}_{11} & \bar{\bar{\Xi}}_{21} & \bar{\bar{\Xi}}_{31} & -\Xi_{41} & * \\ \bar{\bar{\Xi}}_{12} & \bar{\bar{\Xi}}_{22} & 0 & D_1 & -\Xi_{42} \end{bmatrix} < 0 \tag{6.24}$$

where

$$\Xi_{41} = diag\{S \quad R \quad S \quad R \quad S \quad R \quad S \quad R\},\ \Xi_{42} = diag\{S \quad R \quad S \quad R \quad S \quad R \quad I\}.$$

Proof Let $P^{-1} = S$, and $Q^{-1} = R$, we can obtain (6.24) from (6.19). The proof is completed.

Thus, the H_∞ controller design problem is equivalent to the following optimization problem:

$$\min_{P>0,Q>0,S>0,R>0,K,L} \gamma$$

$$\text{subject to (6.24) and } PS = I,\ QR = I.$$

The condition (6.24) is not an LMI, because the matrix P, Q, and their inverse S, R, exist simultaneously. So, we are now in a position to convert (6.24) into an LMI constraint in order to solve the optimization problem by using the MATLAB Toolbox. At this stage, the cone complementarity linearization method [23] is applied. Note that the scalar γ can be included as an optimization variable to obtain a minimum attenuation level. Thus, the observer and controller gains can be readily found under the minimum attenuation level γ by solving the following convex optimization problem.

Problem 6.1 The H_∞ optimal control problem:

$$\min_{P>0,Q>0,S>0,R>0,K,L,\mu>0} tr[P_k S + PS_k + Q_k R + RQ_k + \mu_k]$$

subject to (6.24) where $\gamma^2 = \mu$, and

$$\begin{bmatrix} P & I \\ I & S \end{bmatrix} \geq 0 \tag{6.25}$$

$$\begin{bmatrix} Q & I \\ I & R \end{bmatrix} \geq 0. \tag{6.26}$$

Then the H_∞disturbance-rejection-attenuation level γ can be obtained by $\gamma = \sqrt{\mu}$.

Remark 6.5 The cone implementarity linearization algorithm is proposed in [23], which is widely used in the H_∞ filtering [25] and H_∞control [26] problems. In [25], the value of γ is chosen iteratively until the smallest is obtained. In [26], γ is taken as an optimized variable in the LMI, but is not included in the performance index, so the value of γ obtained is sub-optimal. Because γ can be considered as an optimization value, we add it into the minimum performance index. Then the controller and the observer gains and the minimum attenuation level γ can be got simultaneously as the algorithm is convergent.

6.5 Simulation example

In this section, an uninterruptible power system (UPS) will be taken as an example to demonstrate the effectiveness and applicability of the proposed method. Our objective is to control the PMW inverter, through networks, such that the output AC voltage is kept at the desired setting and undistorted, while maintaining robustness against the disturbances in the load. We consider the UPS with 1KVA. The discrete-time model (6.1) can be obtained with sampling time 10ms at half-load operating point as follows [27]:

$$A=\begin{bmatrix}0.9226 & -0.6330 & 0\\ 1.0 & 0 & 0\\ 0 & 1.0 & 0\end{bmatrix},\ B_2=\begin{bmatrix}1\\0\\0\end{bmatrix},\ B_1=\begin{bmatrix}0.5\\0\\0.2\end{bmatrix},$$

$$C_2=[23.738 \quad 20.287 \quad 0],\ C_1=1,\ D_2=[0.1 \quad 0 \quad 0],\ D_1=1$$

In this example, we want to investigate the effect of random time delays and packet dropouts caused by unreliable communication on the system performances. We assume that the initial conditions are $x_0=[0 \quad 0 \quad 0]^T$, and $\hat{x}_0=[0.5 \quad 0.2 \quad 0.5]^T$and the disturbance input is $w_k=1/k^2$. We choose several values of $\bar{\xi}$, $\bar{\delta}$, $\bar{\alpha}$ and $\bar{\beta}$ to solve the Problem 6.1. In fact, the values of $\bar{\xi},\bar{\delta}$, $\bar{\alpha}$ and $\bar{\beta}$ in a practical network can be obtained by statistics method. The simulation results are shown in Tables 6.1-6.4 where the sensor-to-estimator channel and the controller-to-actuator channel are considered respectively.

First, we take the controller-to-actuator channel into account. In order to analyze the effect of random delays and packet dropouts on the system performance, we let $\bar{\xi}=\bar{\delta}=0.8$, *i.e.*, the random delay rate and packet dropout rate are fixed in the sensor-to-estimator channel, then we study the influence under different packet dropout rate and time delay rate for the controller-to-actuator channel. The results are given in Table 6.1 and Table 6.2, respectively. Table 6.1 gives the comparison of the minimum attenuation level γ_{min} under different packet dropout rate when the one-step time delay rate is 0.2, *i.e.*, $(1-\bar{\alpha})^2\bar{\beta}=0.2$, from which we can see that when the packet dropouts become severer, the minimum value γ_{min} of the H_∞ performance index becomes larger, that is to say, the robustness of the closed-loop networked system to the disturbance input is degraded. Table 6.2 gives the comparison of minimum value γ_{min} under different time delay rate when the random packet dropout rate is 0.3, *i.e.*,

$(1-\bar{\alpha})\bar{\alpha}+(1-\bar{\alpha})^2(1-\bar{\beta})=0.3$. Although the minimum value γ_{min} becomes smaller when the rate of packet received on time becomes larger *i.e.*, the probability of random delay rate becomes smaller, the effect of the time delay rate on the system performance is not as obvious as the effect of random packet dropouts.

Table 6.1 The H_∞ performance comparison of under the different packet dropouts rate for controller-to-actuator channel.

$\bar{\alpha}$	0.1	0.2	0.3	0.4	0.5
P_{drop}	0.7	0.6	0.5	0.4	0.3
γ_{min}	1.1220	1.1192	1.1134	1.1088	1.1060

Table 6.2 The H_∞ performance comparison of under the different time delay rate for controller-to-actuator channel.

$\bar{\alpha}$	0.1	0.2	0.3	0.4	0.5
P_{delay}	0.6	0.5	0.4	0.3	0.2
γ_{min}	1.1057	1.1059	1.1109	1.1109	1.1060

Second, we consider the sensor-to-estimator channel. The random delay rate and packet dropout rate in the controller-to-actuator channel are fixed by $\bar{\alpha}=\bar{\beta}=0.7$. The simulation results are shown in Table 6.3 and Table 6.4. Table 6.3 gives the influence of random dropout rate on the system performance under the one-step delay rate is 0.2, *i.e.*, $(1-\bar{\xi})^2\bar{\delta}=0.2$. It can be easily seen that the H_∞ performance of the closed-loop system becomes worse, *i.e.*, the minimum value γ_{min} is greater when the packet dropout rate becomes larger. Table 6.4 gives the influence of different time delay rate under the packet dropout rate is 0.3, *i.e.*, $(1-\bar{\xi})\bar{\xi}+(1-\bar{\xi})^2(1-\bar{\delta})=0.3$, from which we can see that with the rate of random delays becoming smaller, the minimum attenuation level γ_{min} is also becoming smaller although the change is not obvious.

Table 6.3 The H_∞ performance comparison of under the different packet dropouts rate for sensor-to-estimator channel.

$\bar{\xi}$	0.1	0.2	0.3	0.4	0.5
P_{drop}	0.7	0.6	0.5	0.4	0.3
γ_{min}	1.1183	1.1177	1.1166	1.1152	1.1127

Table 6.4 The H_∞ performance comparison of under the different time delays rate for sensor-to-estimator channel.

$\bar{\xi}$	0.1	0.2	0.3	0.4	0.5
P_{delay}	0.6	0.5	0.4	0.3	0.2
γ_{min}	1.1145	1.1138	1.1136	1.1133	1.1127

Finally, we give the control result under the case of $\bar{\xi} = 0.5$, $\bar{\delta} = 0.5$, $\bar{\alpha} = 0.7$ and $\bar{\beta} = 0.7$. In the sensor-to-estimator channel, the packet dropout rate is 0.375 and the one-step delay rate is 0.125. In the controller-to-actuator channel, the packet dropout rate is 0.237 and the one-step delay rate is 0.063. The values of random variables ξ_k, δ_k, α_k and β_k are shown in Figure 6.1, based on which we give the analysis of data transmission in both channels in Table 6.5.

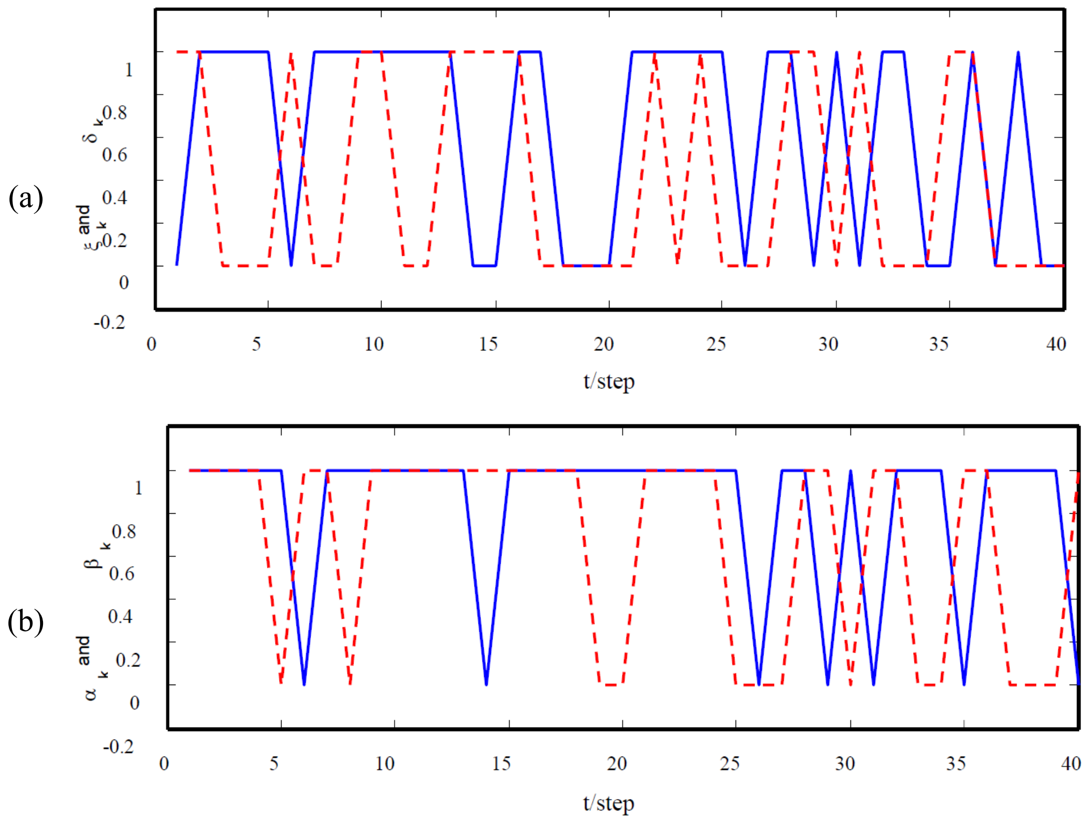

Figure 6.1 The values of random variables under $\bar{\xi} = 0.5$, $\bar{\delta} = 0.5$, and $\bar{\alpha} = 0.7$, $\bar{\beta} = 0.7$. (a) the sensor-to-estimator channel; (b) the controller-to-estimator channel.

Table 6.5 Data Transmission in network.

k	11	12	13	14	15	16	17	18	19
ξ_k	1	1	1	0	0	1	1	0	0
δ_k	0	0	1	1	1	1	0	0	0
y_k	$\tilde{y}_{11}$	$\tilde{y}_{12}$	$\tilde{y}_{13}$	y_{13}	$\tilde{y}_{14}$	$\tilde{y}_{16}$	$\tilde{y}_{17}$	y_{17}	y_{18}
α_k	1	1	1	0	1	1	1	1	1
β_k	1	1	1	1	1	1	1	1	0
u_k	$\tilde{u}_{11}$	$\tilde{u}_{12}$	$\tilde{u}_{13}$	u_{13}	$\tilde{u}_{15}$	$\tilde{u}_{16}$	$\tilde{u}_{17}$	$\tilde{u}_{18}$	$\tilde{u}_{19}$

The minimum attenuation level $\gamma_{min} = 1.1140$ can be obtained by solving Problem 6.1. At the meantime, we can obtain the controller gain and the observer gain as

$$K = [0.0178 \quad -0.1157 \quad 0.0249 \quad 0.0034 \quad 0.0034 \quad 0.0007 \quad 0.0086 \quad -0.0007 \quad 0.0078]$$

$$L = [0.0018 \quad 0.0121 \quad 0.0098 \quad -0.0296 \quad 0.5347 \quad 0.9955 \quad -0.0000 \quad 0.0000 \quad 0.0000]^T$$

The simulation result of the closed-loop state response is shown in Figure 6.2. From the control result, it can be easily seen that the algorithm we proposed is effective.

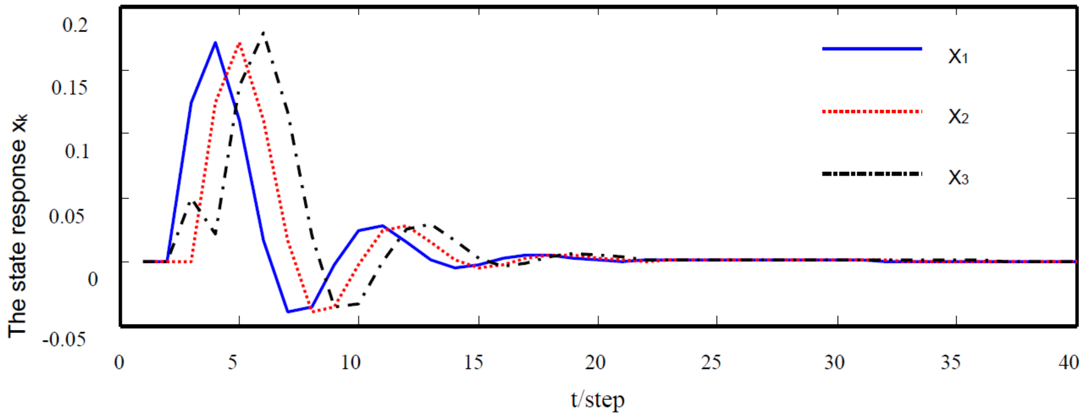

Figure 6.2 The controlled output with $\gamma = 1.1140$ under $\bar{\xi} = 0.5$, $\bar{\delta} = 0.5$, and $\bar{\alpha} = 0.7$, $\bar{\beta} = 0.7$.

6.6 Conclusions

In this chapter, an observer-based control problem has been studied for NCSs with random packet dropouts and communication delays. A novel model has been established to describe the possible one-step transmission delay and multiple packet dropouts from sensors to controllers and from controllers to actuators by employing four Bernoulli distributed variables. By defining some new variables, the original system with delays and packet dropouts can be converted to the augmented system with stochastic parameters. The controller has been designed via an LMI approach to make the closed-loop networked system exponentially mean-square stability and achieve a desired γ disturbance rejection level. Simulation results have demonstrated the feasibility of our control scheme. It is shown that the minimum attenuation level γ is influenced by the rates of random packet dropouts more than the rates of random delays because the time delay, we considered in this chapter is only one step but the packet dropouts may be consecutive. Our future research topics would be the design of controllers for networked systems with multiple packet losses and long random delays.

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Chapter 7 Observer-Based H_∞ Control for Networked Systems with Bounded Random Delays and Consecutive Packet Dropouts

7.1 Introduction

The introduction of communication networks into traditional control systems brings a lot of advantages, such as low cost, simple installation and maintenance, increased system agility, and so on [1]. The systems are the so-called Networked Control Systems (NCSs) which have many industrial applications in traffic, manufacturing plants, remote process and other areas. It is well known that the time delays and packet dropouts are unavoidable phenomena when signals are transmitted through the network. They can degrade the system performance or even cause instability of the controlled systems.

The above important issues have attracted considerable research attention in recent years, see, for example, [2-5] for the time delays and [6-9] for the packet dropouts. In recent literatures [10-12], both time delays and packet dropouts are investigated. The phenomena of time delays and/or packet dropouts happen stochastically in nature. They can be described by Bernoulli distributed white sequence or Markov chains. Various methodologies have been proposed in the literatures for the controller design considering the time delays and/or packet dropouts, such as LQG optimal control [13], H_2 control [14], Robust H_∞ PID control [15], state and output feedback control [16,17], *etc*. As for H_∞-type controllers, the necessary and sufficient LMI conditions for the H_∞ optimal controller are derived in [18] where the packet delivery characteristics of the network are modeled as a Bernoulli sequence or a two-state Markov process. In [19], the communication packet loss is assumed to obey the Bernoulli random binary distribution, and the observer-based feedback controller is designed which makes the networked system robustly exponentially stable in the sense of mean square with a prescribed H_∞ disturbance-rejection-attenuation level. As for the random time delays, the H_∞ controller is designed via an LMI approach in [20]. The two-mode-dependent robust mixed H_2/H_∞ control synthesis is investigated for NCSs in [21] where random delays existing in both forward controller-to-actuator and feedback sensor-to-controller communication links are modeled as Markov chains. The latest references [22,23] consider the control problem for systems with random packet dropouts and time delays simultaneously. An observer-based H_∞ control method is presented in [22] where the controller and observer gains can be obtained by solving certain LMIs. But the model established can only describe the phenomenon of random consecutive packet dropouts or the time delays. Thus, it cannot deal with the random packet dropouts and time delays simultaneously. A new switched model is given in [23] to describe the random packet dropouts and time delays, and a sufficient condition is obtained under which the system is exponentially stable with a desired H_∞ disturbance attenuation level. But the delay considered is assumed to be less than one sampling period.

Motivated by the above-mentioned problem, we study the H_∞ control in this chapter for a class of NCSs with random packet dropouts and time delays which happen in both the Sensor-to-Controller (S-C) channel and the Controller-to-Actuator (C-A) channel, simultaneously. The primary contributions of this chapter lie in the following. (1) A new model is developed by applying a group of Bernoulli distributed stochastic variables to describe the above phenomena where the packet dropout is consecutive even infinite and the time delay is stochastic time-varying and bounded; (2) Different from the augmentation method [22], a full-order observer based control design scheme is proposed. By defining a new Lyapunov functional, a sufficient condition for the existence of the desired controller is derived in terms of LMIs which makes the closed-loop networked control system be asymptotically mean-square stable and achieve the prescribed disturbance attenuation performance.

7.2 Problem formulation

Consider the discrete-time linear stochastic system:

$$\begin{cases} x_{k+1} = Ax_k + B_2 u_k + B_1 w_k \\ \tilde{y}_k = C_2 x_k + C_1 w_k \\ z_k = D_2 x_k + D_1 w_k \end{cases} \tag{7.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $\tilde{y}_k \in \mathbb{R}^r$ is the measured output, $z_k \in \mathbb{R}^m$ is the controlled output, $w_k \in \mathbb{R}^q$ is the exogenous disturbance input belongs to $l_2[0,\infty)$, $u_k \in \mathbb{R}^p$ is the controller received by the actuator, and A, B_1, B_2, C_1, C_2, D_1 and D_2 are known constant matrices with appropriate dimensions.

The NCS we considered in this chapter is shown in Figure 7.1. The data is transmitted through network both from a sensor to an estimator and from a controller to an actuator. Due to the limited bandwidth of the communication channels, there exist possible packet dropouts and time delays. We assume that the random delays are bounded and the packet dropouts are possibly multiple, even possible consecutive infinite. To avoid the network congestion, it is assumed that a packet is only transmitted one time.

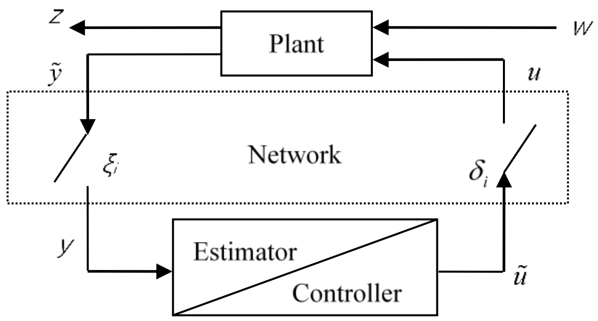


Figure 7.1 Structure for NCS with random delays and packet dropouts.

7.2.1 NCS modeling

Consider the S-C channel firstly. The measurement $y_k \in \mathbb{R}^r$ received by the estimator via an unreliable communication channel can be described by:

$$y_k = \xi_{0,k}\tilde{y}_k + \left(1-\xi_{0,k}\right)\{(1-\xi_{0,k-1})\xi_{1,k}\tilde{y}_{k-1} + [1-\left(1-\xi_{0,k-1}\right)\xi_{1,k}]\{\left(1-\xi_{0,k-2}\right)\left(1-\xi_{1,k-1}\right)\xi_{2,k}\tilde{y}_{k-2} + \cdots + [1-\prod_{i=0}^{d_1-2}\left(1-\xi_{i,k-d_1+i+1}\right)\xi_{d_1-1,k}]\prod_{i=0}^{d_1-1}\left(1-\xi_{i,k-d_1+i}\right)\xi_{d_1,k}\,\tilde{y}_{k-d_1}\}\}\cdots\} \tag{7.2}$$

where $\xi_{i,k}(i = 0,1,\cdots,d_1$ denotes ith variable, and k denotes time instant) are independent identically distributed (i.i.d) Bernoulli random variables which are uncorrelated with each other and satisfy the probabilities $\text{Prob}\{\xi_{i,k} = 1\} = \bar{\xi}_i$ and $\text{Prob}\left\{\xi_{i,k} = 0\right\} = 1 - \bar{\xi}_i$, where $0 \le \bar{\xi}_i \le 1$ and d_1 is the largest time delay of S-C channel.

Similar to the forward measurement channel, the controller u_k received by the actuator from a remote controller can be described by:

$$u_k = \delta_{0,k}\tilde{u}_k + \left(1-\delta_{0,k}\right)\{(1-\delta_{0,k-1})\delta_{1,k}\tilde{u}_{k-1} + [1-\left(1-\delta_{0,k-1}\right)\delta_{1,k}]\{\left(1-\delta_{0,k-2}\right)\left(1-\delta_{1,k-1}\right)\delta_{2,k}\tilde{u}_{k-2} + \cdots + [1-\prod_{i=0}^{d_2-2}\left(1-\delta_{i,k-d_2+i+1}\right)\delta_{d_2-1,k}]\prod_{i=0}^{d_2-1}\left(1-\delta_{i,k-d_2+i}\right)\delta_{d_1,k}\,\tilde{u}_{k-d_2}\}\}\cdots\} \tag{7.3}$$

where $\tilde{u}_k \in \mathbb{R}^p$ is the control input to be designed. $\delta_{i,k}$ $(i = 0,1,\cdots,d_2)$ are also i.i.d Bernoulli distributed random variables which are uncorrelated with other random variables and satisfy probabilities $\text{Prob}\{\delta_{i,k} = 1\} = \bar{\delta}_i$, $\text{Prob}\{\delta_{i,k} = 0\} = 1 - \bar{\delta}_i$, with $0 \le \bar{\delta}_i \le 1$. d_2 is the largest time delay of C-A channel.

Models (7.2) and (7.3) can simultaneously describe consecutive packet dropouts and possible one-step, two-step, up to d_1 and d_2-step random transmission delays in networked systems. They are more general than those proposed in [13,3] where the multiple packet dropouts or random time delays are considered, respectively. Take (7.2) as an example, the following Table 7.1 shows the data transmission case for $d_1 = 2$, *i.e.*,

$$y_k = \xi_{0,k}\tilde{y}_k + (1-\xi_{0,k})\{(1-\xi_{0,k-1})\xi_{1,k}\tilde{y}_{k-1} + [1-(1-\xi_{0,k-1})\xi_{1,k}](1-\xi_{0,k-2})(1-\xi_{1,k-1})\xi_{2,k}\tilde{y}_{k-2}\}$$

It can be readily obtained that the rate of a packet received on time is $\bar{\xi}_0$ the one-step delay rate is $(1-\bar{\xi}_0)^2\bar{\xi}_1$, the two-step delay rate is $(1-\bar{\xi}_0)^2[1-(1-\bar{\xi}_0)\bar{\xi}_1](1-\bar{\xi}_1)\bar{\xi}_2$, and the packet dropout rate is $1-\bar{\xi}_0-(1-\bar{\xi}_0)^2\bar{\xi}_1-(1-\bar{\xi}_0)^2[1-(1-\bar{\xi}_0)\bar{\xi}_1](1-\bar{\xi}_1)\bar{\xi}_2$.

Table 7.1 Data Transmission in network.

k	1	2	3	4	5	6	7	8	9	10
$\xi_{0,k}$	1	1	0	1	0	0	0	0	0	1
$\xi_{1,k}$	-	-	-	0	-	1	0	0	0	-
$\xi_{2,k}$	-	-	-	-	1	-	-	0	0	-
y_k	$\tilde{y}_1$	$\tilde{y}_2$	0	$\tilde{y}_4$	$\tilde{y}_3$	$\tilde{y}_5$	0	0	0	$\tilde{y}_{10}$

Remark 7.1 Most recent work conducts on the systems where the delays exist in the system model and only the missing measurement considered during the data transmission through the network [24,25]. Thus, only the packet dropout of S-C channel is modeled by Bernoulli distributed sequence. In [26], S-C and C-A channels are considered simultaneously where the random delay modeled by Markov chains is only investigated. We model the NCSs by taking the transmission delays and the packet dropouts induced by the network in the S-C and C-A channels into account simultaneously. Moreover, the network transmission delays and packet dropouts may lead to the out of sequence of the arriving packets. This phenomenon is also included in the model we proposed.

To proceed with our discussion, the following definitions should be given.

Let

$$\theta_{j,k}^{(\xi)} = \prod_{i=0}^{j-1}(1-\xi_{i,k+i})\xi_{j,k+j}, \text{ j} = 0,1,\cdots,d_1 \\ \text{with } \theta_{0,k}^{(\xi)} = \xi_{0,k} \tag{7.4}$$

and

$$\theta_{j,k}^{(\delta)} = \prod_{i=0}^{j-1}(1-\delta_{i,k+i})\delta_{j,k+j}, \text{ j} = 1,2,\cdots,d_2 \\ \text{with } \theta_{0,k}^{(\delta)} = \delta_{0,k}. \tag{7.5}$$

Then we have the following statistical properties:

$$E\{\theta_{0,k}^{(\xi)}\} = \bar{\theta}_0^{(\xi)} = \bar{\xi}_0;$$
$$E\{\theta_{j,k}^{(\xi)}\} = \bar{\theta}_j^{(\xi)} = \prod_{i=0}^{j-1}(1-\bar{\xi}_i)\,\bar{\xi}_j, \text{ j} = 1,\cdots,d_1$$
$$E\{\theta_{j,k}^{(\xi)}\theta_{l,k}^{(\xi)}\} = 0, \text{ j} \neq \text{l}$$
$$E\{\theta_{j,k}^{(\xi)}\theta_{l,t}^{(\xi)}\} = \bar{\theta}_j^{(\xi)}\bar{\theta}_l^{(\xi)}, \text{ k} \neq \text{t, j,l} = 0,1,\cdots,d_1.$$

and

$$E\{\theta_{0,k}^{(\delta)}\} = \bar{\theta}_0^{(\delta)} = \bar{\delta}_0;$$
$$E\{\theta_{j,k}^{(\delta)}\} = \bar{\theta}_j^{(\delta)} = \prod_{i=0}^{j-1}(1-\bar{\delta}_i)\,\bar{\delta}_j, \text{ j} = 1,\cdots,d_2$$
$$E\{\theta_{j,k}^{(\delta)}\theta_{l,k}^{(\delta)}\} = 0, \text{ j} \neq \text{l}$$
$$E\{\theta_{j,k}^{(\delta)}\theta_{l,t}^{(\delta)}\} = \bar{\theta}_j^{(\delta)}\bar{\theta}_l^{(\delta)}, \text{ k} \neq \text{t, j,l} = 0,1,\cdots,d_2.$$

7.2.2 Observer-based controller design

Consider the full-order dynamic observer-based feedback controller of the following structure:

$$\begin{cases} \hat{x}_{k+1} = A_c\hat{x}_k + B_c y_k \\ \tilde{u}_k = C_c\hat{x}_k \end{cases} \tag{7.6}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the observer state, and the feedback gains A_c, B_c, C_c are to be designed.

Denote $\eta_k = [x_k^T \quad \hat{x}_k^T]^T$. Then the closed-loop system of (7.1) with (7.2)-(7.3) is described by

$$\begin{cases} \eta_{k+1} = \bar{A}\eta_k + \sum_{i=1}^{d}\bar{\bar{A}}_i\eta_{k-i} + \sum_{i=0}^{d_1}\hat{A}_i\eta_{k-i} + \sum_{i=0}^{d_2}\breve{A}_i\eta_{k-i} + \bar{\Gamma}w_k + \sum_{i=1}^{d_1}\bar{\bar{\Gamma}}_i w_{k-i} + \sum_{i=0}^{d_1}\hat{\Gamma}_i w_{k-i} \\ z_k = \bar{D}\eta_k + D_1 w_k \end{cases} \tag{7.7}$$

where

$$\bar{A} = \begin{bmatrix} A & \bar{\beta}_0 B_2 C_c \\ \bar{\alpha}_0 B_c C_2 & A_c \end{bmatrix},\ \bar{\bar{A}}_i = \begin{bmatrix} 0 & \bar{\beta}_i B_2 C_c \\ \bar{\alpha}_i B_c C_2 & 0 \end{bmatrix},\ \hat{A}_i = \begin{bmatrix} 0 & 0 \\ \tilde{\alpha}_{i,k} B_c C_2 & 0 \end{bmatrix},\ \breve{A}_i = \begin{bmatrix} 0 & \tilde{\beta}_{i,k} B_2 C_c \\ 0 & 0 \end{bmatrix},$$

$$\bar{D} = [D_2 \quad 0],\ \bar{\Gamma} = \begin{bmatrix} B_1 \\ \bar{\alpha}_0 B_c C_1 \end{bmatrix},\ \bar{\bar{\Gamma}}_i = \begin{bmatrix} 0 \\ \bar{\alpha}_i B_c C_1 \end{bmatrix},\ \hat{\Gamma}_i = \begin{bmatrix} 0 \\ \tilde{\alpha}_{i,k} B_c C_1 \end{bmatrix},\ \tilde{\alpha}_{i,k} = \alpha_{i,k} - \bar{\alpha}_i,\ \tilde{\beta}_{i,k} = \beta_{i,k} - \bar{\beta}_i$$

$$\alpha_{i,k} = \prod_{j=0}^{i-1}(1-\theta_{j,k-j}^{(\xi)})\,\theta_{i,k-i}^{(\xi)},\ \beta_{i,k} = \prod_{j=0}^{i-1}(1-\theta_{j,k-j}^{(\delta)})\,\theta_{i,k-i}^{(\delta)}.$$

$\bar{\alpha}_i = E\{\alpha_{i,k}\}$, $\bar{\beta}_i = E\{\beta_{i,k}\}$ are given in (7.4) and (7.5) where $\theta_{j,k}^{(\xi)}(\mathrm{j} = 0,\cdots,d_1)$ and $\theta_{j,k}^{(\delta)}(\mathrm{j} = 0,\cdots,d_2)$ are replaced by their expectations $\bar{\theta}_j^{(\xi)}(\mathrm{j} = 0,\cdots,d_1)$ and $\bar{\theta}_j^{(\delta)}(\mathrm{j} = 0,\cdots,d_2)$, $d = \max\{d_1,d_2\}$.

For any d_1 and d_2, there are three cases:

(i) When $d_1 > d_2$, *i.e.*, the transmission delay of S-C channel is larger than that of C-A channel, the parameter matrices in (7.7) for $i = d_2+1,\cdots,d$ are given as $\bar{\bar{A}}_i = \begin{bmatrix} 0 & 0 \\ \bar{\alpha}_i B_c C_2 & 0 \end{bmatrix}$ and $\breve{A}_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(ii) When $d_1 < d_2$, *i.e.*, the transmission delay of S-C channel is smaller than that of C-A channel, the parameter matrices in (7.7) for $i = d_1+1,\cdots,d$ are given as $\bar{\bar{A}}_i = \begin{bmatrix} 0 & \bar{\beta}_i B_2 C_c \\ 0 & 0 \end{bmatrix}$, $\hat{A}_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{\bar{\Gamma}}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\hat{\Gamma}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(iii) When $d_1 = d_2$, *i.e.*, the transmission delays of S-C channel and C-A channel are the same, the parameter matrices are defined in (7.7).

In the following discussion, we assume that $d_1 = d_2$ without loss of generality. The other two cases can be obtained directly as a special case.

Remark 7.2 In the practical engineering, the state of the controlled system may not be fully measurable, so we depend on the measured output to reconstruct the complete state information and design the observer-based controller. There is another technique to deal with the control problem when the state is not measurable, *i.e.*, the output feedback controller design method (e.g., [27]).

Up to now, the original system is parameterized by stochastic parameters $\alpha_{i,k}$ and $\beta_{i,k}$. For the derivation of our main results, the following statistical properties are useful.

$$E\{(\alpha_{i,k}-\bar{\alpha}_i)^2\} = \bar{\alpha}_i(1-\bar{\alpha}_i);\ \mathrm{i} = 0,\cdots,d_1,$$
$$E\{(\beta_{i,k}-\bar{\beta}_i)^2\} = \bar{\beta}_i(1-\bar{\beta}_i);\ \mathrm{i} = 0,\cdots,d_2,$$
$$E\{(\beta_{i,k}-\bar{\beta}_i)(\beta_{j,k}-\bar{\beta}_j)\} = -\bar{\beta}_i\bar{\beta}_j;\ \mathrm{i} \neq \mathrm{j},$$
$$E\{(\alpha_{i,k}-\bar{\alpha}_i)(\alpha_{j,k}-\bar{\alpha}_j)\} = -\bar{\alpha}_i\bar{\alpha}_j;\ \mathrm{i} \neq \mathrm{j},$$
$$E\{(\alpha_{i,k}-\bar{\alpha}_i)(\beta_{j,k}-\bar{\beta}_j)\} = 0.$$

Our objective is to design a controller in the form of (7.6) for the networked-based system (7.1), such that for all possible missing measurements and transmission delays which exist both in forward S-C and backward C-A channels, the closed-loop system satisfies the following two requirements simultaneously:

(Q1) The closed-loop system (7.7) is asymptotically mean-square stable.

(Q2) Under zero initial condition, the controlled output z_k satisfies

$$\sum_{k=0}^{\infty} E\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\left\{\sum_{j=0}^{d}\|w_{k-j}\|^2\right\} \tag{7.8}$$

for all nonzero w_k, where $\gamma > 0$ is a prescribed scalar.

7.3 Controller performance analysis

In this section, we establish a sufficient condition guarantees the asymptotically stable in the sense of mean square with an H_∞ disturbance attenuation level for given controller gains A_c, B_c and C_c of system (7.7), which will be the fundamental of the H_∞ controller design in the next section.

Lemma 7.1 [28] Let $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^n$ and matrix $Q > 0$. Then, we have $x^TQy + y^TQx \leq x^TQx + y^TQy$.

Theorem 7.1 Given a scalar $\gamma > 0$ and the controller matrices A_c, B_c and C_c. The closed-loop system (7.7) is asymptotically mean-square stable and the controlled output z_k satisfies (7.8) if there exist matrices $P > 0$, $Q_j > 0\ (j = 1, \cdots, d)$, $R > 0$ and S satisfying

$$\begin{bmatrix} \bar{\Omega} + SW + (SW)^T & * & * & * & * & * & * & * & * \\ P\bar{\Psi} & -P & * & * & * & * & * & * & * \\ \bar{P}\hat{\Psi}_1 & 0 & -\bar{P} & * & * & * & * & * & * \\ \bar{P}\hat{\Psi}_2 & 0 & 0 & -\bar{P} & * & * & * & * & * \\ \sqrt{d}R\bar{\Xi} & 0 & 0 & 0 & -R & * & * & * & * \\ \sqrt{d}\bar{R}\hat{\Xi}_1 & 0 & 0 & 0 & 0 & -\bar{R} & * & * & * \\ \sqrt{d}\bar{R}\hat{\Xi}_2 & 0 & 0 & 0 & 0 & 0 & -\bar{R} & * & * \\ \sqrt{d}S^T & 0 & 0 & 0 & 0 & 0 & 0 & -R & * \\ \hat{D} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{7.9}$$

where

$$\Omega = diag\{-P + \textstyle\sum_{j=1}^{d} Q_j, -Q_1, -Q_2, \cdots, -Q_d\},$$

$$\bar{\Omega} = diag\{\Omega, -I_{d+1} \otimes \gamma^2 I_q\},$$

$$W = [I_{2n} \quad 0_{2n\times n(d-1)} \quad -I_{2n} \quad 0_{2n\times q(d+1)}],$$

$$\bar{P} = I_{d+1} \otimes P,\ \bar{R} = I_{d+1} \otimes R,$$

$$\bar{\Psi} = [\bar{A} \quad \bar{\bar{A}}_1 \quad \cdots \quad \bar{\bar{A}}_d \quad \bar{\Gamma} \quad \bar{\bar{\Gamma}}_1 \quad \cdots \quad \bar{\bar{\Gamma}}_d],$$

$$\bar{\Xi} = [\sqrt{d}(\bar{A} - I) \quad \sqrt{d}\bar{\bar{A}}_1 \quad \cdots \quad \sqrt{d}\bar{\bar{A}}_d \quad \sqrt{d}\bar{\Gamma} \quad \sqrt{d}\bar{\bar{\Gamma}}_1 \quad \cdots \quad \sqrt{d}\bar{\bar{\Gamma}}_d],$$

$$\hat{\Psi}_1 = \hat{\Xi}_1 = diag\{\sqrt{\sigma_0}\hat{A}_1, \hat{Z}_1, \sqrt{\sigma_0}\hat{\Gamma}_1, \hat{N}_1\},$$

$$\hat{\Psi}_2 = \hat{\Xi}_2 = diag\{\sqrt{\rho_0}\hat{A}_2, \hat{Z}_2, I_{d+1} \otimes 0_{2n\times q}\},$$

$$\sigma_i = 2(d_1 + 1)\bar{\alpha}_i(1 - \bar{\alpha}_i),\ \rho_i = (d_2 + 1)\bar{\beta}_i(1 - \bar{\beta}_i),$$

$$\hat{Z}_1 = diag\{\sqrt{\sigma_1}, \sqrt{\sigma_2}, \cdots, \sqrt{\sigma_d}\} \otimes \hat{A}_1,$$

$$\hat{Z}_2 = diag\{\sqrt{\rho_1}, \sqrt{\rho_2}, \cdots, \sqrt{\rho_d}\} \otimes \hat{A}_2,$$

$$\hat{N}_1 = diag\{\sqrt{\sigma_1}, \sqrt{\sigma_2}, \cdots, \sqrt{\sigma_d}\} \otimes \hat{\Gamma}_1,$$

$$\hat{D} = \begin{bmatrix} \bar{D} & 0_{m\times nd} & D_1 & 0_{m\times qd} \end{bmatrix},$$

$$\hat{A}_1 = \begin{bmatrix} 0 & 0 \\ B_c C_2 & 0 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} 0 & B_2 C_c \\ 0 & 0 \end{bmatrix}, \hat{\Gamma}_1 = \begin{bmatrix} 0 \\ B_c C_1 \end{bmatrix},$$

Proof Denote

$$\Theta_k = [\eta_k^T \quad \bar{\eta}_k^T]^T, \ \ \bar{\eta}_k = [\eta_{k-1}^T \quad \cdots \quad \eta_{k-d}^T]^T.$$

Choose the Lyapunov-Krasovskii functional for system (7.7) as

$$\mathrm{V}_k(\Theta_k) = \mathrm{V}_{1k}(\Theta_k) + \mathrm{V}_{2k}(\Theta_k) + \mathrm{V}_{3k}(\Theta_k) \tag{7.10}$$

where

$$\mathrm{V}_{1k}(\Theta_k) = \eta_k^T P \eta_k \tag{7.11}$$

$$\mathrm{V}_{2k}(\Theta_k) = \sum_{j=1}^{d} \sum_{i=k-j}^{k-1} \eta_i^T Q_j \eta_i \tag{7.12}$$

$$\mathrm{V}_{3k}(\Theta_k) = \textstyle\sum_{j=-d}^{-1} \sum_{i=k+j}^{k-1} \varepsilon_i^T R \varepsilon_i, \ \varepsilon_k = \eta_{k+1} - \eta_k \tag{7.13}$$

Notice that

$$E\{\breve{A}_i^T P \hat{A}_j\} = 0,$$

$$E\{\breve{A}_i^T P \hat{\Gamma}_j\} = 0, (i, j = 0,1,\cdots,d),$$

$$E\{\hat{A}_i^T P \hat{A}_i\} = \bar{\alpha}_i(1-\bar{\alpha}_i)\hat{A}_1^T P \hat{A}_1,$$

$$E\{\breve{A}_i^T P \breve{A}_i\} = \bar{\beta}_i(1-\bar{\beta}_i)\hat{A}_2^T P \hat{A}_2,$$

$$E\{\hat{\Gamma}_i^T P \hat{\Gamma}_i\} = \bar{\alpha}_i(1-\bar{\alpha}_i)\hat{\Gamma}_1^T P \hat{\Gamma}_1, \ (i = 0,1,\cdots,d)$$

where $\hat{A}_1$, $\hat{A}_2$ and $\hat{\Gamma}_1$ defined by Theorem 7.1.

From Lemma 7.1, we have

$$\eta_{k-i}^T \hat{A}_i^T P \hat{A}_j \eta_{k-j} + \eta_{k-j}^T \hat{A}_j^T P \hat{A}_i \eta_{k-i} \le \eta_{k-i}^T \hat{A}_i^T P \hat{A}_i \eta_{k-i} + \eta_{k-j}^T \hat{A}_j^T P \hat{A}_j \eta_{k-j}, \tag{7.14}$$

$$\eta_{k-i}^T \breve{A}_i^T P \breve{A}_j \eta_{k-j} + \eta_{k-j}^T \breve{A}_j^T P \breve{A}_i \eta_{k-i} \le \eta_{k-i}^T \breve{A}_i^T P \breve{A}_i \eta_{k-i} + \eta_{k-j}^T \breve{A}_j^T P \breve{A}_j \eta_{k-j}, \tag{7.15}$$

$$w_{k-i}^T \hat{\Gamma}_i^T P \hat{A}_j \eta_{k-j} + w_{k-j}^T \hat{\Gamma}_j^T P \hat{A}_i \eta_{k-i} \le w_{k-i}^T \hat{\Gamma}_i^T P \hat{\Gamma}_i w_{k-i} + \eta_{k-j}^T \hat{A}_j^T P \hat{A}_j \eta_{k-j}, \tag{7.16}$$

Then, calculating the difference of V_k along the trajectory of system (7.7), we have

$$E\{\Delta \mathrm{V}_{1k}\} = E\{V_{1(k+1)}(\Theta_{k+1})|\Theta_k\} - V_{1k}(\Theta_k) = E\{\eta_{k+1}^T P \eta_{k+1} - \eta_k^T P \eta_k\}$$

$$\begin{aligned}
&= E\Big\{\big[\bar{A}\eta_k + \textstyle\sum_{i=1}^{d} \bar{\bar{A}}_i \eta_{k-i} + \sum_{i=0}^{d_1} \hat{A}_i \eta_{k-i} + \sum_{i=0}^{d_2} \breve{A}_i \eta_{k-i} + \bar{\Gamma} w_k + \sum_{i=1}^{d_1} \bar{\bar{\Gamma}}_i w_{k-i} + \sum_{i=0}^{d_1} \hat{\Gamma}_i w_{k-i}\big]^T P \\
&\times \left[\bar{A}\eta_k + \sum_{i=1}^{d} \bar{\bar{A}}_i \eta_{k-i} + \sum_{i=0}^{d_1} \hat{A}_i \eta_{k-i} + \sum_{i=0}^{d_2} \breve{A}_i \eta_{k-i} + \bar{\Gamma} w_k + \sum_{i=1}^{d_1} \bar{\bar{\Gamma}}_i w_{k-i} + \sum_{i=0}^{d_1} \hat{\Gamma}_i w_{k-i}\right]\Bigg\} - \eta_k^T P \eta_k \\
&= \eta_k^T(-P + \bar{A}^T P \bar{A})\eta_k + \sum_{i=1}^{d} 2\eta_{k-i}^T \bar{\bar{A}}_i^T P \bar{A} \eta_k + \sum_{i=1}^{d}\sum_{j=1}^{d} 2\,\eta_{k-i}^T \bar{\bar{A}}_i^T P \bar{\bar{A}}_j \eta_{k-j} \\
&\quad + \sum_{i=0}^{d_1} \sigma_i \eta_{k-i}^T \hat{A}_1^T P \hat{A}_1 \eta_{k-i}
\end{aligned} \tag{7.17}$$

$$+\sum_{i=0}^{d_2}\rho_i\eta_{k-i}^T\hat{A}_2^TP\hat{A}_2\eta_{k-i}+w_k^T\bar{\Gamma}^TP\bar{\Gamma}w_k+\sum_{i=1}^{d}2w_{k-i}^T\bar{\bar{\Gamma}}_i^TP\bar{\Gamma}w_k$$

$$+\sum_{i=1}^{d_1}\sum_{j=1}^{d_1}2\,w_{k-i}^T\bar{\bar{\Gamma}}_i^TP\bar{\bar{\Gamma}}_jw_{k-i}+\sum_{i=0}^{d_1}\sigma_iw_{k-i}^T\hat{\Gamma}_1^TP\hat{\Gamma}_1w_{k-i}+2w_k^T\bar{\Gamma}^TP\bar{\mathrm{A}}\eta_k$$

$$+\sum_{i=1}^{d}2w_k^T\bar{\Gamma}^TP\bar{\bar{A}}_i\eta_{k-i}+\sum_{i=1}^{d}2w_{k-i}^T\bar{\Gamma}_i^TP\bar{A}\eta_k$$

$$+\sum_{i=1}^{d}\sum_{j=1}^{d}2\,w_{k-i}^T\bar{\bar{\Gamma}}_i^TP\bar{\bar{\mathrm{A}}}_i\eta_{k-i}=\varsigma_k^T\Pi\varsigma_k$$

where

$$\varsigma_k=[\eta_k^T\quad\bar{\eta}_k^T\quad w_k^T\quad\bar{w}_k^T]^T,\ \bar{w}_k=[w_{k-1}^T\quad\cdots\quad w_{k-d}^T]^T$$

$$\Pi=\begin{bmatrix}\Pi_{11}&*&*&*\\\Pi_{21}&\Pi_{22}&*&*\\\Pi_{31}&\Pi_{32}&\Pi_{33}&*\\\Pi_{41}&\Pi_{42}&\Pi_{43}&\Pi_{44}\end{bmatrix},$$

$$\Pi_{11}=-P+\bar{A}^TP\bar{A}+\sigma_0\hat{A}_1^TP\hat{A}_1+\rho_0\hat{A}_2^TP\hat{A}_2,$$

$$\Pi_{21}=\bar{Z}^TP\bar{A},\ \Pi_{22}=\bar{Z}^TP\bar{Z}+\hat{Z}_1^T(I_d\otimes P)\hat{Z}_1+\hat{Z}_2^T(I_d\otimes P)\hat{Z}_2,$$

$$\Pi_3=\bar{\Gamma}^TP\bar{A},\ \Pi_{32}=\bar{\Gamma}^TP\bar{Z},\ \Pi_{33}=\bar{\Gamma}^TP\bar{\Gamma}+\sigma_0\hat{\Gamma}_1^TP\hat{\Gamma}_1,$$

$$\Pi_4=\bar{\Gamma}^TP\bar{A}$$

$$\Pi_{41}=\bar{\mathrm{N}}^TP\bar{A},\ \Pi_{42}=\bar{\mathrm{N}}^TP\bar{Z},\ \Pi_{43}=\bar{\mathrm{N}}^TP\bar{\Gamma},\ \Pi_{44}=\bar{\mathrm{N}}^TP\bar{N}+\hat{N}_1^T(I_d\otimes P)\hat{N}_1,$$

$$\bar{Z}=\begin{bmatrix}\bar{\bar{A}}_1&\bar{\bar{A}}_2\cdots&\bar{\bar{A}}_d\end{bmatrix},\ \bar{N}=\begin{bmatrix}\bar{\bar{\Gamma}}_1&\bar{\bar{\Gamma}}_2\cdots&\bar{\bar{\Gamma}}_d\end{bmatrix},$$

with $\hat{Z}_1,\ \hat{Z}_2,\ \hat{N}_1,\hat{A}_1,\ \hat{A}_2$ and $\hat{\Gamma}_1$ are defined by Theorem 7.1.

Next, it can be derived that

$$E\{\Delta V_{2k}\}=\sum_{j=1}^{d}(\eta_k^TQ_j\eta_k-\eta_{k-j}^TQ_j\eta_{k-j})\tag{7.18}$$

$$E\{\Delta V_{3k}\}=E\{\varepsilon_k^TdR\varepsilon_k-\sum_{i=k-d}^{k-1}\varepsilon_i^TR\varepsilon_i\},\tag{7.19}$$

According to the definition of ε_k, we have

$$\varepsilon_k=(\bar{A}-I)\eta_k+\sum_{i=1}^{d}\bar{\bar{A}}_i\eta_{k-i}+\sum_{i=0}^{d_1}\hat{A}_i\eta_{k-i}+\sum_{i=0}^{d_2}\breve{A}_i\eta_{k-i}+\bar{\Gamma}w_k+\sum_{i=1}^{d_1}\bar{\bar{\Gamma}}_iw_{k-i}+\sum_{i=0}^{d_1}\hat{\Gamma}_iw_{k-i}\tag{7.20}$$

Then, for any matrix S, the following equation always holds:

$$2\varsigma_k\,S\left[\eta_k-\eta_{k-d}-\sum_{i=k-d}^{k-1}\varepsilon_i\right]=0\tag{7.21}$$

Thus, we can obtain that

$$E\{\Delta V_{3k}\}=E\{\varepsilon_k^TdR\varepsilon_k-\sum_{i=k-d}^{k-1}\varepsilon_i^TR\varepsilon_i\}$$

$$=E\{\varepsilon_k^TdR\varepsilon_k\}+\varsigma_k\,dSR^{-1}S^T\varsigma_k+\varsigma_k\,S(\eta_k-\eta_{k-d})+(\eta_k-\eta_{k-d})^TS^T\varsigma_k\tag{7.22}$$

$$-\sum_{i=k-d}^{k-1}(R\varepsilon_i+S^T\varsigma_k)^TR^{-1}(R\varepsilon_i+S^T\varsigma_k)$$

Substituting (7.20) into (7.22), and considering the relation (7.17), we have

$$
\begin{aligned}
E\{\Delta V_k\} &= E\{\Delta V_{1k}\} + E\{\Delta V_{2k}\} + E\{\Delta V_{3k}\} \\
&\le \varsigma_k \{\Omega + SW + (SW)^T + \overline{\Psi}^T P \overline{\Psi} + \widehat{\Psi}_1^T \bar{P} \widehat{\Psi}_1 + \widehat{\Psi}_2^T \bar{P} \widehat{\Psi}_2 + d \bar{\Xi}^T R \bar{\Xi} \\
&\quad + d \hat{\Xi}_1^T R \hat{\Xi}_1 + d \hat{\Xi}_2^T R \hat{\Xi}_2 + d S R^{-1} S^T) \varsigma_k
\end{aligned} \tag{7.23}
$$

where Ω, W, $\overline{\Psi}$, $\widehat{\Psi}_1$, $\widehat{\Psi}_2$, $\bar{\Xi}$, $\hat{\Xi}_1$, $\hat{\Xi}_2$, $\bar{P}$ and $\bar{R}$ are defined by Theorem 7.1.

To analyze the H_∞ performance of the closed-loop system (7.7), we introduce the following index:

$$
\begin{aligned}
J &= \sum_{k=0}^{\infty} E\left\{ z_k^T z_k - \gamma^2 \sum_{i=0}^{d} w_{k-i}^T w_{k-i} \right\} \\
&= \sum_{k=0}^{\infty} E\left\{ z_k^T z_k - \gamma^2 \sum_{i=0}^{d} w_{k-i}^T w_{k-i} + \Delta V_k \right\} + E\{V_0\} - E\{V_\infty\} \\
&\le \sum_{k=0}^{\infty} E\left\{ z_k^T z_k - \gamma^2 \sum_{i=0}^{d} w_{k-i}^T w_{k-i} + \Delta V_k \right\}
\end{aligned} \tag{7.24}
$$

From (7.23), we have

$$
\begin{aligned}
&E\{z_k^T z_k - \gamma^2 w_k^T w_k + \Delta V_k\} \\
&\le \varsigma_k^T \{\overline{\Omega} + SW + (SW)^T + \overline{\Psi}^T P \overline{\Psi} + \widehat{\Psi}_1^T P \widehat{\Psi}_1 + \widehat{\Psi}_2^T P \widehat{\Psi}_2 + d \bar{\Xi}^T R \bar{\Xi} + d \hat{\Xi}_1^T P \hat{\Xi}_1 \\
&\quad + d \hat{\Xi}_2^T P \hat{\Xi}_2 + d S R^{-1} S^T + \widehat{D}^T \widehat{D}\} \varsigma_k
\end{aligned} \tag{7.25}
$$

By Schur complement, it follows from (7.9) that $\{z_k^T z_k - \gamma^2 w_k^T w_k + \Delta V_k\} < 0$, which implies that $J < 0$. Therefore, the condition (7.8) holds for all nonzero w_k. Similar to the above derivation, we can obtain that the forward difference of V_k satisfies $\Delta V_k < 0$ in the absence of w_k, which indicates the closed-loop system (7.7) is asymptotically mean-square stable [29]. This completes the proof.

7.4 H_∞ controller design

In this section, we propose a design approach of H_∞ controller based on the result obtained in the last section with the consideration of random consecutive packet dropouts and bounded time delays, simultaneously. The main results are given by the following Theorems.

Theorem 7.2 Consider the closed-loop system (7.7) with a prescribed H_∞ performance index $\gamma > 0$. The system (7.7) is asymptotically mean-square stable with an H_∞ disturbance attenuation level γ if there exist matrices $P > 0$, $Q_j > 0$ $(j = 1,2,\cdots,\mathrm{d})$, $R > 0$, S and H such that the following inequality holds:

$$\begin{bmatrix} \bar{\Omega}+SW+(SW)^T & * & * & * & * & * & * & * & * \\ H\bar{\Psi} & \tilde{P} & * & * & * & * & * & * & * \\ \hat{H}\hat{\Psi}_1 & 0 & \hat{P} & * & * & * & * & * & * \\ \hat{H}\hat{\Psi}_2 & 0 & 0 & \hat{P} & * & * & * & * & * \\ \sqrt{d}H\bar{\Xi} & 0 & 0 & 0 & \tilde{R} & * & * & * & * \\ \sqrt{d}\hat{H}\hat{\Xi}_1 & 0 & 0 & 0 & 0 & \hat{R} & * & * & * \\ \sqrt{d}\hat{H}\hat{\Xi}_2 & 0 & 0 & 0 & 0 & 0 & \hat{R} & * & * \\ \sqrt{d}S^T & 0 & 0 & 0 & 0 & 0 & 0 & -R & * \\ \hat{D} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{7.26}$$

where

$$\tilde{P} = P - H - H^T,\ \tilde{R} = R - H - H^T,\ \hat{P} = I_{d+1} \otimes \tilde{P},\ \hat{R} = I_{d+1} \otimes \tilde{R},\ \hat{H} = I_{d+1} \otimes H$$

$\bar{\Omega}$, $\bar{W}$, $\bar{\Psi}$, $\hat{\Psi}_1, \hat{\Psi}_2$, $\bar{\Xi}, \hat{\Xi}_1$, $\hat{\Xi}_2$ and $\hat{D}$ are defined by Theorem 7.1.

Proof Using the fact $P - H - H^T \geq -HP^{-1}H^T = H_p$, $R - H - H^T \geq -HR^{-1}H^T = H_R$, and by defining $\hat{H}_p = I_{d+1} \otimes H_p$, $\hat{H}_R = I_{d+1} \otimes H_R$, we can obtain that

$$\begin{bmatrix} \bar{\Omega}+SW+(SW)^T & * & * & * & * & * & * & * & * \\ H\bar{\Psi} & H_p & * & * & * & * & * & * & * \\ \hat{H}\hat{\Psi}_1 & 0 & \hat{H}_p & * & * & * & * & * & * \\ \hat{H}\hat{\Psi}_2 & 0 & 0 & \hat{H}_p & * & * & * & * & * \\ \sqrt{d}H\bar{\Xi} & 0 & 0 & 0 & H_R & * & * & * & * \\ \sqrt{d}\hat{H}\hat{\Xi}_1 & 0 & 0 & 0 & 0 & \hat{H}_R & * & * & * \\ \sqrt{d}\hat{H}\hat{\Xi}_2 & 0 & 0 & 0 & 0 & 0 & \hat{H}_R & * & * \\ \sqrt{d}S^T & 0 & 0 & 0 & 0 & 0 & 0 & -R & * \\ \hat{D} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{7.27}$$

Then, performing a congruence transformation $J = diag\{I, \underbrace{PH^{-1}, \cdots, PH^{-1}}_{2d+3}, \underbrace{RH^{-1}, \cdots, RH^{-1}}_{2d+3}, I, I\}$ to (7.27), the inequality in (7.9) is obtained. Thus, the proof is completed.

Theorem 7.3 Consider the closed-loop system (7.7) with a prescribed H_∞ performance index $\gamma > 0$. The system (7.7) is asymptotically mean-square stable with an H_∞ disturbance attenuation level γ if there exist matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, $Q_{j1} > 0$, $Q_{j2} > 0$, $Q_{j3} > 0$ (j $= 1,2,\cdots,$ d), $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, S, H_1, H_2, H_3, A_c, B_c and C_c such that the following inequality holds:

$$\begin{bmatrix} \Lambda_1 & * & * & * & * & * & * & * & * \\ \Lambda_2 & \Lambda_6 & * & * & * & * & * & * & * \\ \Lambda_3 & 0 & \hat{\Lambda}_6 & * & * & * & * & * & * \\ \Lambda_4 & 0 & 0 & \hat{\Lambda}_6 & * & * & * & * & * \\ \sqrt{d}\Lambda_5 & 0 & 0 & 0 & \Lambda_7 & * & * & * & * \\ \sqrt{d}\Lambda_3 & 0 & 0 & 0 & 0 & \hat{\Lambda}_7 & * & * & * \\ \sqrt{d}\Lambda_4 & 0 & 0 & 0 & 0 & 0 & \hat{\Lambda}_7 & * & * \\ \sqrt{d}S^T & 0 & 0 & 0 & 0 & 0 & 0 & -\Lambda_8 & * \\ \hat{D} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{7.28}$$

where

$$\Lambda_1 = \bar{\Omega} + SW + (SW)^T,$$

$$P = \begin{bmatrix} P_1 & * \\ P_2 & P_3 \end{bmatrix},\ Q_j = \begin{bmatrix} Q_{j1} & * \\ Q_{j2} & Q_{j3} \end{bmatrix}, j = 1,2,\cdots,d$$

$$\Lambda_2 = [\Lambda_{21} \quad \Lambda_{22} \quad \Lambda_{23} \quad \Lambda_{24}],$$

$$\Lambda_{21} = \begin{bmatrix} H_1A + \bar{\alpha}_0 H_2 B_c C_2 & \bar{\beta}_0 H_1 B_2 C_c + H_2 A_c \\ H_3A + \bar{\alpha}_0 H_2 B_c C_2 & \bar{\beta}_0 H_3 B_2 C_c + H_2 A_c \end{bmatrix},$$

$$\Lambda_{22} = \begin{bmatrix} \bar{\alpha}_1 H_2 B_c C_2 & \bar{\beta}_1 H_1 B_2 C_c & \bar{\alpha}_1 H_2 B_c C_2 & \bar{\beta}_2 H_1 B_2 C_c & \cdots & \cdots & \bar{\alpha}_d H_2 B_c C_2 & \bar{\beta}_d H_1 B_2 C_c \\ \bar{\alpha}_1 H_2 B_c C_2 & \bar{\beta}_1 H_3 B_2 C_c & \bar{\alpha}_1 H_2 B_c C_2 & \bar{\beta}_2 H_3 B_2 C_c & \cdots & \cdots & \bar{\alpha}_d H_2 B_c C_2 & \bar{\beta}_d H_3 B_2 C_c \end{bmatrix},$$

$$\Lambda_{23} = \begin{bmatrix} H_1 B_1 + \bar{\alpha}_0 H_2 B_c C_1 \\ H_3 B_1 + \bar{\alpha}_0 H_2 B_c C_1 \end{bmatrix},\ \Lambda_{24} = \begin{bmatrix} \bar{\alpha}_1 H_2 B_c C_1 & \bar{\alpha}_2 H_2 B_c C_1 & \cdots & \bar{\alpha}_d H_2 B_c C_1 \\ \bar{\alpha}_1 H_2 B_c C_1 & \bar{\alpha}_2 H_2 B_c C_1 & \cdots & \bar{\alpha}_d H_2 B_c C_1 \end{bmatrix}$$

$$\Lambda_3 = diag\{\Lambda_{31}, \Lambda_{32}\}, \qquad \Lambda_4 = diag\{\Lambda_{41}, I_d \otimes 0_{2n\times q}\},$$

$$\Lambda_{31} = diag\{\sqrt{\sigma_0} \quad \sqrt{\sigma_1} \quad \cdots \quad \sqrt{\sigma_d}\} \otimes \tilde{\Lambda}_{31},$$

$$\Lambda_{32} = diag\{\sqrt{\sigma_0} \quad \sqrt{\sigma_1} \quad \cdots \quad \sqrt{\sigma_d}\} \otimes \tilde{\Lambda}_{32},$$

$$\Lambda_{41} = diag\{\sqrt{\rho_0} \quad \sqrt{\rho_1} \quad \cdots \quad \sqrt{\rho_d}\} \otimes \tilde{\Lambda}_{41},$$

$$\tilde{\Lambda}_{31} = \begin{bmatrix} H_2 B_c C_2 & 0_{n\times n} \\ H_2 B_c C_2 & 0_{n\times n} \end{bmatrix},\ \tilde{\Lambda}_{31} = \begin{bmatrix} H_2 B_c C_1 \\ H_2 B_c C_1 \end{bmatrix},\ \tilde{\Lambda}_{41} = \begin{bmatrix} 0_{n\times n} & H_1 B_2 C_c \\ 0_{n\times n} & H_3 B_2 C_c \end{bmatrix},$$

$$\Lambda_5 = [\Lambda_{51} \quad \Lambda_{52} \quad \Lambda_{53} \quad \Lambda_{54}],$$

$$\Lambda_{51} = \begin{bmatrix} H_1(A - I) + \bar{\alpha}_0 H_2 B_c C_2 & \bar{\beta}_0 H_1 B_2 C_c + H_2(A_c - I) \\ H_3(A - I) + \bar{\alpha}_0 H_2 B_c C_2 & \bar{\beta}_0 H_3 B_2 C_c + H_2(A_c - I) \end{bmatrix},$$

$$\Lambda_{52} = \Lambda_{22},\ \Lambda_{53} = \Lambda_{23},\ \Lambda_{54} = \Lambda_{24},$$

$$\Lambda_6 = \begin{bmatrix} P_1 - H_1 - H_1^T & * \\ P_2 - H_3 - H_2^T & P_3 - H_2 - H_2^T \end{bmatrix},\ \Lambda_7 = \begin{bmatrix} R_1 - H_1 - H_1^T & * \\ R_2 - H_3 - H_2^T & R_3 - H_2 - H_2^T \end{bmatrix},$$

$$\hat{\Lambda}_6 = I_d \otimes \Lambda_6,\ \hat{\Lambda}_7 = I_d \otimes \Lambda_7,\ \Lambda_8 = \begin{bmatrix} R_1 & * \\ R_2 & R_3 \end{bmatrix}.$$

Proof Let us partition H as

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_2 \end{bmatrix} \tag{7.29}$$

where H_1 and H_2 are all nonsingular without loss of generality. Furthermore, partition P and R as

$$P = \begin{bmatrix} P_1 & * \\ P_2 & P_3 \end{bmatrix},\ R = \begin{bmatrix} R_1 & * \\ R_2 & R_3 \end{bmatrix} \tag{7.30}$$

Then substituting (7.29)-(7.30) into (7.26), we can get (7.28) immediately.

Observe that the controller gain matrices A_c, B_c and C_c are coupled in the inequality (7.28). There are some ways to transform the nonlinear matrix inequality into LMI one, such as matrix patitioning method [25] and singular value decomposition technique [30]. Here, we adopt the method developed in [27] to convert (7.28) into an LMI for designed parameters A_c, B_c and C_c. Assume that B_2 is of full column rank which can be satisfied for many practical systems (e.g., [31,32]). We use $\mathrm{T} \in \mathbb{R}^{n\times n}$ to denote the corresponding invertible matrix, *i.e.*, $TB_2 = \begin{bmatrix} I_{p\times p} \\ 0_{(n-p)\times p} \end{bmatrix}$. In addition, we define a constant matrix $M_{(n-p)\times p}$ as follows:

$$M_{(n-p)\times p} = \begin{cases} [I_{(n-p)\times(n-p)} \quad 0_{(n-p)\times(2p-n)}] & n \le 2p \\ \begin{bmatrix} I_{p\times p} \\ 0_{(n-2p)\times p} \end{bmatrix} & n > 2p \end{cases}$$

Remark 7.3 For each B_2 of full column rank, the corresponding T is not unique, such as a special form of

$$T = \begin{bmatrix} (B_2^T B_2)^{-1} B_2^T \\ B_2^{\perp} \end{bmatrix},$$

where $B_2^{\perp}$ represents an orthogonal basis for the null space of B_2.

Theorem 7.4 Consider the closed-loop system (7.7) with a prescribed H_∞ performance index $\gamma > 0$. Given the constants λ_1, λ_2 and λ_3, the system (7.7) is asymptotically mean-square stable with an H_∞ disturbance attenuation level γ if there exist matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, $Q_{j1} > 0$, $Q_{j2} > 0$, $Q_{j3} > 0$ ($j = 1,2,\cdots,d$), $R_1 > 0$, $R_2 > 0$, $R_3 > 0$, $S, \tilde{A}_c$, $\tilde{B}_c$, $\tilde{C}_c$, $H_1 = \begin{bmatrix} H_{11} & H_{12} \\ \lambda_1 M_{(n-p)\times p} H_{11} & H_{22} \end{bmatrix} T$, H_2 and $H_3 = \begin{bmatrix} \lambda_2 H_{11} & H_{32} \\ \lambda_3 M_{(n-p)\times p} H_{11} & H_{42} \end{bmatrix} T$ such that the following LMI holds:

$$\begin{bmatrix} \Lambda_1 & * & * & * & * & * & * & * & * \\ \Lambda_2 & \Lambda_6 & * & * & * & * & * & * & * \\ \overline{\Lambda}_3 & 0 & \Lambda_6 & * & * & * & * & * & * \\ \overline{\Lambda}_4 & 0 & 0 & \Lambda_6 & * & * & * & * & * \\ \sqrt{d}\overline{\Lambda}_5 & 0 & 0 & 0 & \Lambda_7 & * & * & * & * \\ \sqrt{d}\overline{\Lambda}_3 & 0 & 0 & 0 & 0 & \Lambda_7 & * & * & * \\ \sqrt{d}\overline{\Lambda}_4 & 0 & 0 & 0 & 0 & 0 & \Lambda_7 & * & * \\ \sqrt{d}S^T & 0 & 0 & 0 & 0 & 0 & 0 & -\Lambda_8 & * \\ \widehat{D} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0 \tag{7.31}$$

where

$$\overline{\Lambda}_2 = [\overline{\Lambda}_{21} \quad \overline{\Lambda}_{22} \quad \overline{\Lambda}_{23} \quad \overline{\Lambda}_{24}],$$

$$\overline{\Lambda}_{21} = \begin{bmatrix} H_1 A + \bar{\alpha}_0 \tilde{B}_c C_2 & \bar{\beta}_0 \Sigma_1 \tilde{C}_c + \tilde{A}_c \\ H_3 A + \bar{\alpha}_0 \tilde{B}_c C_2 & \bar{\beta}_0 \Sigma_3 \tilde{C}_c + \tilde{A}_c \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} I \\ \lambda_1 M_{(n-p)\times p} I \end{bmatrix}, \ \Sigma_3 = \begin{bmatrix} \lambda_2 I \\ \lambda_3 M_{(n-p)\times p} I \end{bmatrix},$$

$$\overline{\Lambda}_{22} = \begin{bmatrix} \bar{\alpha}_1 \tilde{B}_c C_2 & \bar{\beta}_1 \Sigma_1 \tilde{C}_c & \bar{\alpha}_1 \tilde{B}_c C_2 & \bar{\beta}_2 \Sigma_1 \tilde{C}_c & \cdots & \cdots & \bar{\alpha}_d \tilde{B}_c C_2 & \bar{\beta}_d \Sigma_1 \tilde{C}_c \\ \bar{\alpha}_1 \tilde{B}_c C_2 & \bar{\beta}_1 \Sigma_3 \tilde{C}_c & \bar{\alpha}_1 \tilde{B}_c C_2 & \bar{\beta}_2 \Sigma_3 \tilde{C}_c & \cdots & \cdots & \bar{\alpha}_d \tilde{B}_c C_2 & \bar{\beta}_d \Sigma_3 \tilde{C}_c \end{bmatrix},$$

$$\overline{\Lambda}_{23} = \begin{bmatrix} H_1 B_1 + \bar{\alpha}_0 \tilde{B}_c C_1 \\ H_3 B_1 + \bar{\alpha}_0 \tilde{B}_c C_1 \end{bmatrix}, \ \overline{\Lambda}_{24} = \begin{bmatrix} \bar{\alpha}_1 \tilde{B}_c C_1 & \bar{\alpha}_2 \tilde{B}_c C_1 & \cdots & \bar{\alpha}_d \tilde{B}_c C_1 \\ \bar{\alpha}_1 \tilde{B}_c C_1 & \bar{\alpha}_2 \tilde{B}_c C_1 & \cdots & \bar{\alpha}_d \tilde{B}_c C_1 \end{bmatrix},$$

$$\overline{\Lambda}_3 = diag\{\overline{\Lambda}_{31}, \overline{\Lambda}_{32}\}, \ \overline{\Lambda}_4 = diag\{\overline{\Lambda}_{41}, I_d \otimes 0_{2n\times q}\},$$

$$\overline{\Lambda}_{31} = diag\{\sqrt{\sigma_0} \quad \sqrt{\sigma_1} \quad \cdots \quad \sqrt{\sigma_d}\} \otimes \widehat{\Lambda}_{31},$$

$$\overline{\Lambda}_{32} = diag\{\sqrt{\sigma_0} \quad \sqrt{\sigma_1} \quad \cdots \quad \sqrt{\sigma_d}\} \otimes \widehat{\Lambda}_{32},$$

$$\overline{\Lambda}_{41} = diag\{\sqrt{\rho_0} \quad \sqrt{\rho_1} \quad \cdots \quad \sqrt{\rho_d}\} \otimes \widehat{\Lambda}_{41},$$

$$\widehat{\Lambda}_{31}=\begin{bmatrix}\tilde{B}_cC_2 & 0_{n\times n}\\ \tilde{B}_cC_2 & 0_{n\times n}\end{bmatrix},\ \widehat{\Lambda}_{32}=\begin{bmatrix}\tilde{B}_cC_1\\ \tilde{B}_cC_1\end{bmatrix},\ \widehat{\Lambda}_{41}=\begin{bmatrix}0_{n\times n} & \Sigma_1\tilde{C}_c\\ 0_{n\times n} & \Sigma_3\tilde{C}_c\end{bmatrix},$$

$$\overline{\Lambda}_5=[\overline{\Lambda}_{51}\quad \overline{\Lambda}_{52}\quad \overline{\Lambda}_{53}\quad \overline{\Lambda}_{54}],$$

$$\overline{\Lambda}_{51}=\begin{bmatrix}H_1(A-I)+\bar{\alpha}_0\tilde{B}_cC_2 & \bar{\beta}_0\Sigma_1\tilde{C}_c+\tilde{A}_c-H_2\\ H_3(A-I)+\bar{\alpha}_0\tilde{B}_cC_2 & \bar{\beta}_0\Sigma_3\tilde{C}_c+\tilde{A}_c-H_2\end{bmatrix},$$

$$\overline{\Lambda}_{52}=\overline{\Lambda}_{22},\ \overline{\Lambda}_{53}=\overline{\Lambda}_{23},\ \overline{\Lambda}_{54}=\overline{\Lambda}_{24}$$

Furthermore, the controller parameter can be designed as

$$A_c=H_2^{-1}\tilde{A}_c,\ B_c=H_2^{-1}\tilde{B}_c,\ C_c=H_{11}^{-1}\tilde{C}_c. \tag{7.32}$$

Proof Define the new variables $\tilde{A}_c=H_2A_c$, $\tilde{B}_c=H_2B_c$ and $\tilde{C}_c=H_{11}C_c$. We can obtain (7.31) directly from (7.28). Since H_1 and H_2 are all nonsingular, the inverse of $H_{11}$and H_2 exist. This completes the proof.

Thus, the H_∞ controller design problem is equivalent to the following optimization problem:

Problem 7.1 $\min\limits_{\substack{P_i>0,Q_{ji}>0,R_i>0,S,H_i,\tilde{A}_c,\tilde{B}_c,\tilde{C}_c\\(i=1,2,3;j=1,2,\cdots,d)}}\gamma$ subject to (7.31) where $\gamma^2=\mu$

7.5 Simulation research

In this section, we will present two practical examples to demonstrate the effectiveness and applicability of the proposed method. The first example is High Incidence Research Model (HIRM) aircraft system which is unstable and provided to illustrate the stabilization and effectiveness for the designed controller. The second one is a stirred tank system which is a multivariable system and used to demonstrate the robustness of the controller designed.

Example 1 (High Incidence Research Model (HIRM)) We consider the longitudinal dynamics of the High Incidence Research Model (HIRM) aircraft. A linearization of the nonlinear simulation [33] about Mach number 0.3 and an altitude of 5000 ft has been used as the basis of the design. A discretized representation based on a sample interval of 0.025 s is as follows [14]:

$$A=\begin{bmatrix}0.9862 & 0.0243 & 0\\ -0.0264 & 0.9894 & 0\\ -0.0003 & 0.0249 & 1.0000\end{bmatrix},\ B_2=\begin{bmatrix}-0.0038\\ -0.0810\\ -0.0010\end{bmatrix},\ B_1=\begin{bmatrix}0.1\\ 0.1\\ 0.2\end{bmatrix},$$

$$C_2=[1\quad 0\quad 1],\ C_1=1,\ D_2=[1\quad 0.5\quad 1],\ D_1=1,$$

We assume that the disturbance input is $w_k=1/k^2$. It can be seen that the system is unstable. Our aim is to design an observer-based controller such that the closed-loop system is asymptotically stable and achieves the obtained minimum guaranteed H_∞

performance when the network characteristics are taken into account. Assume that random delay bounds of both channels are $d_1 = d_2 = 2$, *i.e.*, the one-step delay and two-step delay will occur randomly as well as consecutive packet dropouts during the data transmission. Without loss of generality, we first give the control result under the case of $\alpha_0 = 0.9$, $\beta_0 = 0.9$, $\alpha_1 = 0.2$, $\beta_1 = 0.2$, $\alpha_2 = 0.5$, and $\beta_2 = 0.5$. In each channel, the rate of a packet received on time is 0.9, the one-step delay rate is 0.002, the two-step delay rate is 0.00392, and the packet dropout rate is 0.09408. The minimum attenuation level $\gamma_{min} = 0.8530$ can be obtained by solving Problem 7.1. At the same time, the controller gain matrices can be obtained as

$$A_c = \begin{bmatrix} 0.7941 & 0.1604 & -0.0290 \\ -0.0338 & 0.8371 & -0.0101 \\ -0.0452 & 0.0759 & 0.7775 \end{bmatrix}, B_c = \begin{bmatrix} -0.0834 \\ -0.0878 \\ -0.1902 \end{bmatrix},$$

$$C_c = [-0.0362 \quad -0.0560 \quad 0.0003]$$

The initial conditions are $x_0 = [-0.17 \quad 0.2 \quad -0.3]^T$ and $\hat{x}_0 = [0 \quad 0 \quad 0]^T$. The simulation result of the closed-loop state response is shown in Figure 7.2. Moreover, by calculation, we can get $\frac{\sum_{k=0}^{\infty} E\{\|z_k\|^2\}}{\sum_{k=0}^{\infty} E\{\|w_k\|^2+\|w_{k-1}\|^2+\|w_{k-2}\|^2\}} = 0.5482 < \gamma^2 = 0.7276$. Hence, the effectiveness of the H_∞ controller design can be illustrated.

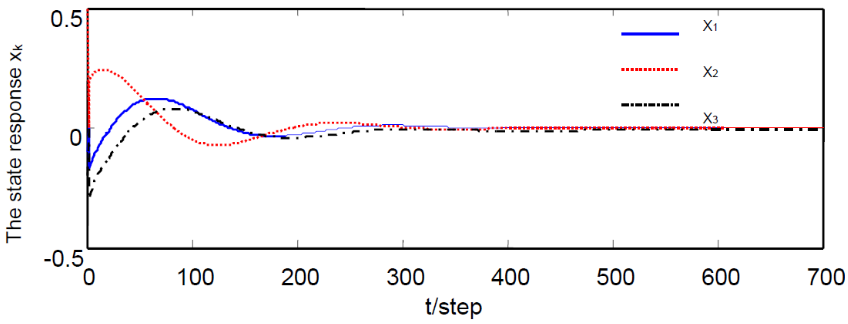


Figure 7.2 The closed-loop state response with $\gamma = 0.8530$ under $\alpha_0 = 0.9$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$ and $\beta_0 = 0.9$, $\beta_1 = 0.2$, $\beta_2 = 0.5$.

We also give the control result under the case of $\alpha_0 = 0.4$, $\beta_0 = 0.4$, $\alpha_1 = 0.7$, $\beta_1 = 0.7$, $\alpha_2 = 0.8$, and $\beta_2 = 0.8$. In each channel, the rate of a packet received on time is 0.4, the one-step delay rate is 0.2520, the two-step delay rate is 0.0501, and the packet dropout rate is 0.2979. The minimum attenuation level $\gamma_{min} = 1.4841$ can be obtained by solving Problem 7.1 which is larger than case 1 because the rate of a packet received on time is lower. At the same time, we can obtain the controller gains as

$$A_c = \begin{bmatrix} 0.6955 & 0.2410 & -0.0380 \\ 0.0115 & 0.9641 & -0.0598 \\ 0.0448 & 0.7179 & 0.5136 \end{bmatrix}, B_c = \begin{bmatrix} -0.0265 \\ -0.0239 \\ -0.0571 \end{bmatrix},$$

$$C_c = [-0.0035 \quad -0.0031 \quad -0.0004]$$

The initial conditions are $x_0 = [0.44 \quad 0.12 \quad 0.25]^T$ and $\hat{x}_0 = [0 \quad 0 \quad 0]^T$. The simulation result of the closed-loop state response is shown in Figure 7.3. We also have that $\frac{\sum_{k=0}^{\infty} E\{\|z_k\|^2\}}{\sum_{k=0}^{\infty} E\{\|w_k\|^2+\|w_{k-1}\|^2+\|w_{k-2}\|^2\}} = 2.0492 < \gamma^2 = 2.2026$.

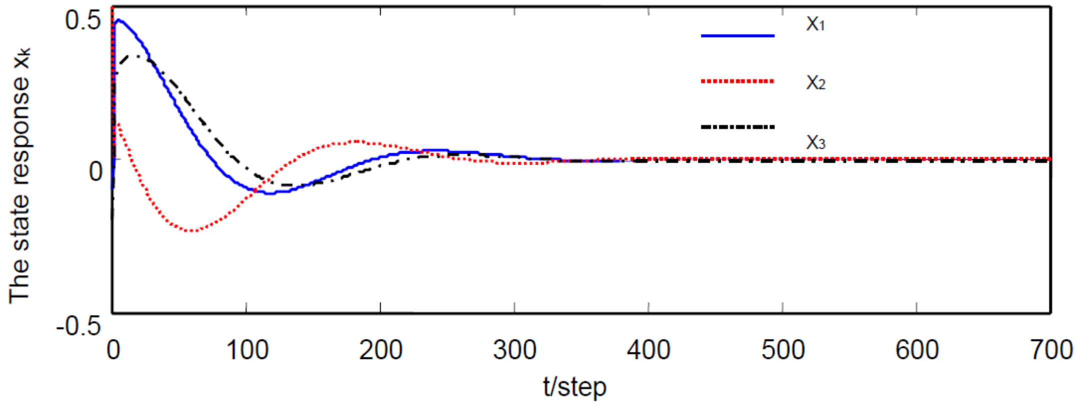


Figure 7.3 The closed-loop state response with $\gamma = 1.4841$ under $\alpha_0 = 0.4$, $\alpha_1 = 0.7$, $\alpha_2 = 0.8$ and $\beta_0 = 0.4$, $\beta_1 = 0.7$, $\beta_2 = 0.8$.

It can be easily seen that the proposed algorithm is effective to stabilize the unstable plant subject to time-varying network-induced delays and randomly missing packets.

Furthermore, we consider the case of $\alpha_0 = 0.4$, $\beta_0 = 0.4$, $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 = 0$, and $\beta_2 = 0$, in order to compare with the result of [14] where only the packet dropouts are considered. In this case, the on-time rates, one-step delay rates and the packet dropout rates of both channels are 0.4, 0.36 and 0.24, respectively. We can obtain the following controller gain matrices as

$$A_c = \begin{bmatrix} 0.5715 & 1.0165 & -0.3856 \\ 0.0098 & 1.6654 & -0.4277 \\ 0.0328 & 2.3925 & -0.3695 \end{bmatrix}, \; B_c = \begin{bmatrix} -0.0420 \\ -0.0400 \\ -0.0876 \end{bmatrix},$$

$$C_c = [-0.0035 \quad -0.0061 \quad -0.0003]$$

The minimum disturbance attenuation level $\gamma_{min} = 0.7874$ can be obtained simultaneously. However, when the rates of the packet received on time are 0.4 for both channels, the minimum attenuation level γ_{min} is 1.2573 by using the control method in [14] although where the latest received data is used when the packets are lost during the data transmission. Moreover, due to the state augmentation method applied in [14], the dimensions of the controller gain matrices are larger than that of this chapter. From the comparison with [14], we can see that the controller design scheme proposed in this chapter has the advantages of smaller minimum disturbance attenuation level and smaller dimensions of the controller gain matrices.

Example 2 (a stirred tank system) In this example, we take a multivariable system into consideration. The discretized version of a stirred tank [15] with the sampling period 0.1 s is described by:

$$A = \begin{bmatrix} 0.9512 & 0 & 0 & 0 \\ 0 & 0.9048 & 0.0670 & 0.0226 \\ 0 & 0 & 0.8825 & 0 \\ 0 & 0 & 0 & 0.9048 \end{bmatrix},\ B_2 = \begin{bmatrix} 4.8771 & 4.8771 \\ -10.1895 & 3.5686 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T$$

and we choose

$$B_1 = [1 \quad 0 \quad 1 \quad 0]^T,\ C_1 = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix},\ D_2 = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

In [15], this system is also taken as an example to demonstrate the effectiveness of the proposed H_∞ PID controller. By considering the influence of network, the transmission delays are only considered for both channels, but the packet dropouts are not taken into account in [15]. Furthermore, the value of γ is given as 80 which cannot ensure the optimality. In this example, the maximum network-induced delays are assumed to be the same as [15], *i.e.*, $d_1 = 3$ and $d_2 = 2$, that is, the bounds of random time delays for the S-C channel and the C-A channel are 3 and 2, respectively. The noise w_k is assumed as a zero mean Gaussian white sequence with the variance 0.001 and there will be an exogenous disturbance $f = 1/k^2$ when $k = 200$. Under the case of $\alpha_0 = 0.7$, $\alpha_1 = 0.6$, $\alpha_2 = 0.9$, $\alpha_3 = 0.1$, $\beta_0 = 0.8$,, $\beta_1 = 0.3$ and $\beta_2 = 0.1$, we can obtain the optimal H_∞ performance as $\gamma_{min} = 5.3728$ by solving Problem 7.1 which is great smaller than that of [15]. The controller gains can also be obtained simultaneously as follows.

$$A_c = \begin{bmatrix} 0.7886 & 0.0281 & 0.0681 & -0.0775 \\ 0.0283 & 0.8428 & 0.3250 & -0.2735 \\ 0.0499 & 0.0575 & 0.7977 & -0.0139 \\ 0.0531 & 0.0595 & 0.0584 & 0.7358 \end{bmatrix},\ B_c = \begin{bmatrix} -0.0319 & -0.0878 \\ -0.0214 & -0.1845 \\ -0.0018 & -0.1231 \\ 0.0022 & -0.1340 \end{bmatrix},$$

$$C_c = \begin{bmatrix} -0.0028 & 0.0019 & 0.0008 & -0.0009 \\ 0.0015 & -0.0028 & -0.0049 & 0.0032 \end{bmatrix}$$

The simulation results are shown in Figure 7.4 from which it can be easily seen that the first output has better robustness to the disturbance than the second one. The values of random variables $\xi_{0,k}$, $\xi_{1,k}$, $\xi_{2,k}$, $\xi_{3,k}$ and $\delta_{0,k}$, $\delta_{1,k}$, $\delta_{2,k}$ are given in Figure 7.5 from $k = 0$ to $k = 50$. According to which we can compute the data transmission case. In addition, if we can obtain the on-time rates, packet dropout rates and random time delay rates of both channels in practice by statistic method, we can set the expectation values of random variables $\xi_{0,k}$, $\xi_{1,k}$, $\xi_{2,k}$, $\xi_{3,k}$ and $\delta_{0,k}$, $\delta_{1,k}$, $\delta_{2,k}$. This is another advantage of our method proposed depend on probability.

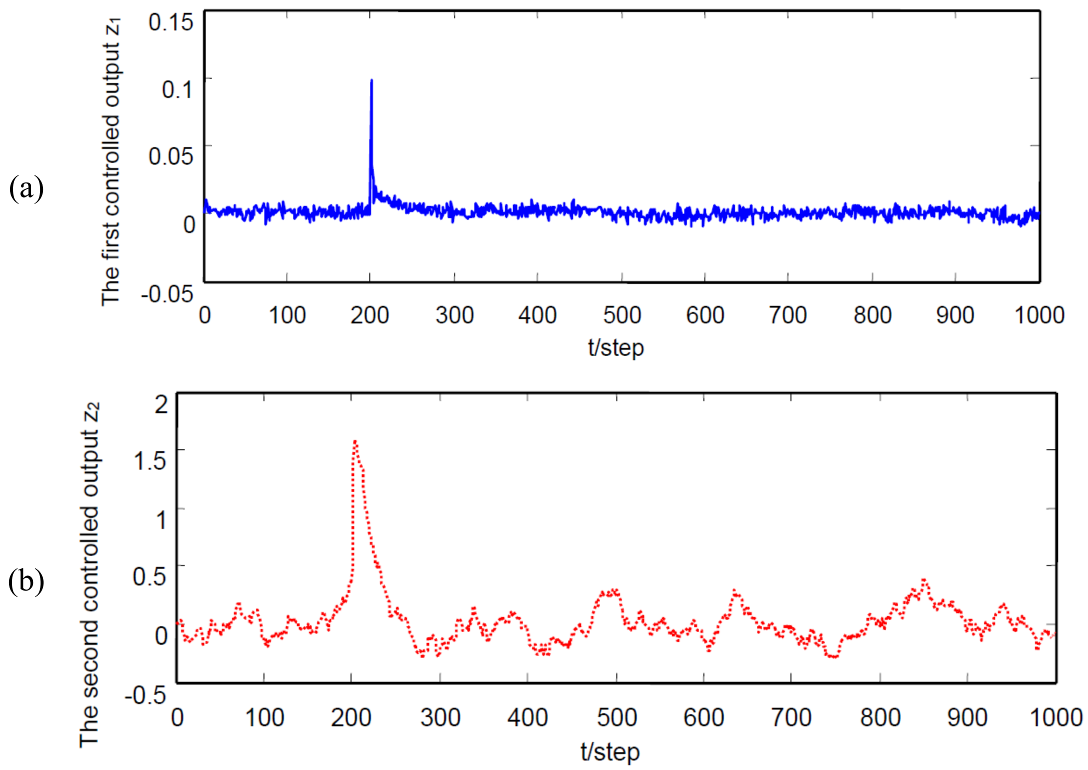
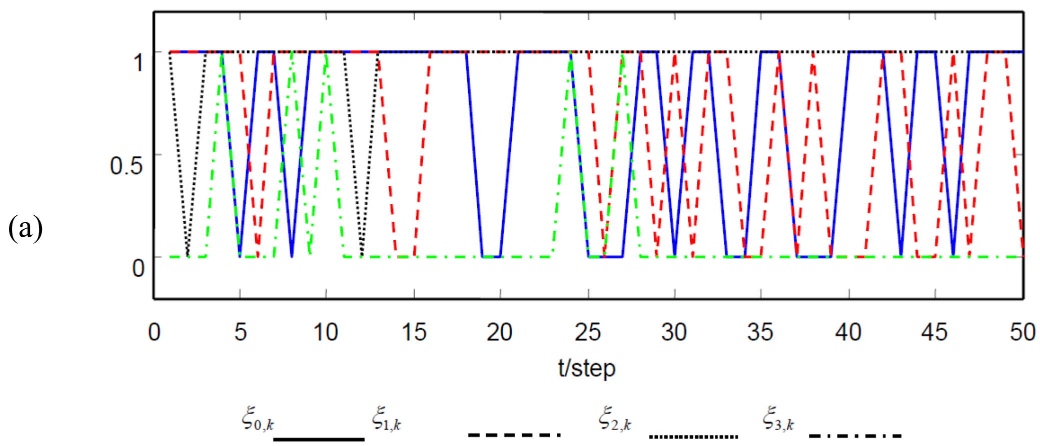


Figure 7.4 The controlled output z_k under $\alpha_0 = 0.7, \alpha_1 = 0.6, \alpha_2 = 0.9,\ \alpha_3 = 0.1$ and $\beta_0 = 0.8,\ \beta_1 = 0.3, \beta_2 = 0.1$. (a) The first component of controlled output z_k; (b) the second component of controlled output z_k.



(b)

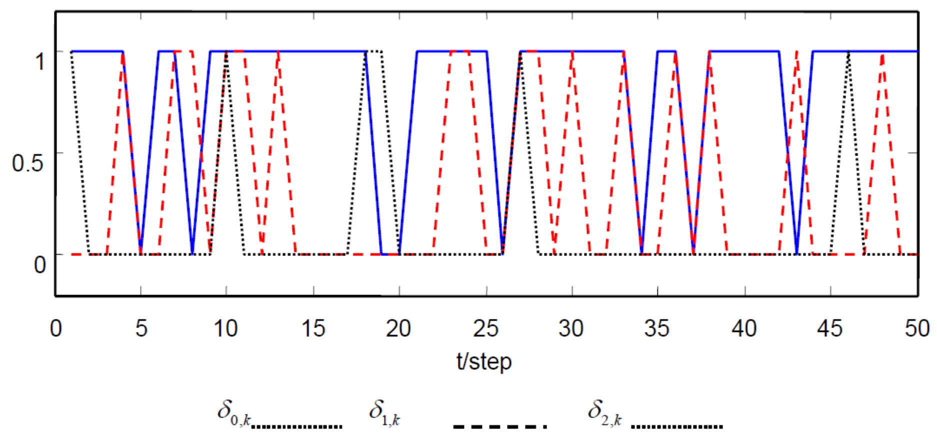


Figure 7.5 The values of random variables under $\alpha_0 = 0.7, \alpha_1 = 0.6, \alpha_2 = 0.9, \alpha_3 = 0.1$ and $\beta_0 = 0.8, \beta_1 = 0.3, \beta_2 = 0.1$. (a) The sensor-to-controller channel; (b) The controller-to-actuator channel.

7.6 Conclusions

In this chapter, a full order observer-based H_∞ control problem for NCSs has been investigated. First, we have modeled an NCS by considering the random transmission delays and packet dropouts from the sensor to the controller and from the controller to the actuator. A few of Bernoulli distributed sequences are employed to describe the possibly consecutive packet dropouts and bounded delays. Furthermore, in order to avoid augmenting the state, we have converted the original system with random delays and packet dropouts into the stochastic parameterized system with delayed states by defining some new variables. Second, sufficient conditions for the asymptotically mean-square stability of the stochastic system are established. Based on the LMI technique, we have developed the design method for the H_∞ controller. Moreover, the stabilization and effectiveness for the proposed controller have been demonstrated via two practical examples in simulation. In future work, we are going to consider the out of sequence of the arriving packets during the data transmission in the controller design.

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Chapter 8 H_∞ Control for Networked Stochastic Nonlinear Systems with Randomly Occurring Sensor Saturations, Multiple Delays and Packet Dropouts

8.1 Introduction

The rapid development of communication network has a deep effect on the traditional control systems structure and the networked control system (NCS) has emerged in the meantime where a network link is involved to transmit the data packet among the spatially distributed sensor, controller and actuator. Nowadays, NCS has been and continuous to be a rather hot topic due to its widely applications in many areas such as car suspension system, spacecraft, manufacturing plants, self-motion robots, and remote medical service [1]. Since the communication capacity of a network is typically limited, the phenomena of packet dropouts [2], out-of-order packets [3], time delays [4], *etc*. will occur unavoidably during the data transmission. These network-induced challenging problems will degrade the system performance or even lead the system to be unstable because of their random changes in complex network environments. Thus, one of the key issues is how to utilize the imperfect information to handle the stability and stabilization of the system. Many researchers have devoted themselves to investigating the control problem for such a system, e.g., LQG control [5], output tracking control [6], H_∞ control [7], finite-horizon reliable control [8], and so on. In practical engineering applications, the time-stamp technique [9,10] is often employed to know whether a packet is received or not and a finite memory buffer is also used at the receiver side for a finite maximum delay. The existence of transmission delays will result in multiple data packets arriving at the buffer within one sampling period. Compared with the existing studies, including the aforementioned references, the researches have been done in the context of using the multiple packets not one packet for estimation problem, because it is an intuitive idea that the estimation performance will be improved if multiple packets are used for estimation update at each moment [11,12]. Besides, the case of multiple packets arrived at the receiver side can be met with multiple radios scenario. To support real-time applications such as multimedia and emergency services, the capacity of single radio may severely limit the Qos for such traffic, multiple radios can overcome this restriction and provide additional links to support better Qos mechanisms [13]. To the best of the authors' knowledge, the control problems for NCS by using multiple packets have not been fully investigated yet. This is the first motivation of our current study.

On the other hand, almost all real-world systems are influenced by certain nonlinear disturbances and the stability analysis and synthesis for nonlinear control systems has been a main stream of research for several decades [14]. The nonlinear property of systems can be depicted by a T-S fuzzy model which has been proven to be a conceptually simple and powerful tool to approximate a complex nonlinear system [15]. The other nonlinear characteristic is reflected by the state nonlinearity of the system which is generally represented by sector-bound nonlinearity because it can cover Lipschitz and norm-bounded conditions as the special cases [16]. Moreover, in today's network environments, the nonlinearities may occur in a random way due probably to the random fluctuation of network loads and the unreliability of wireless links. Such network-induced nonlinearities are customarily referred to as randomly occurring nonlinearities (RONs), see e.g., [17,18] and the references therein.

In addition, the physical sensors, as an essential part of NCSs, inevitably show nonlinear characteristics under harsh environments. This type of nonlinearity is called the sensor saturation. For example, when the input signal of differential pressure sensor is beyond the amplifier's output range, the sensor will suffer the saturation phenomenon. The sector-nonlinearity is also used to represent sensor saturation as well as quantization and has been extensively studied in the existing literatures, see e.g., [19-21]. Furthermore, the network-induced sensor saturations often occur randomly because of physical limitations of system components as well as the difficulties in ensuring high fidelity and timely arrival of the sensing signals through a possibly unreliable network with limited bandwidths. Neglecting this nonlinear phenomenon will make the system performance dissatisfactory. On the basis of well-studied results by assuming the linearity of sensors, the randomly occurring sensor saturation (ROSS) has been drawing an increasing research interest [22-24].

Motivated by the above discussion, this chapter focuses on the H_∞ control problem for networked stochastic nonlinear systems subject to random transmission delays, packet losses as well as randomly occurring sensor saturations. The main contributions of this chapter are summarized as follows:

(1) A comprehensive structure of NCS with RON is developed in which the sensor outputs subject to ROSS may encounter random delays and packet dropouts during the transmission through the network. Meanwhile, a new model is adopted to describe multiple packets arrived at the buffer in the receiver side during one sampling period because of the existence of transmission delays.

(2) A new approach to design the H_∞ controller based on a nonlinear observer is proposed by using the multiple packets in the buffer, such that the dynamics of the closed-loop system is guaranteed to be asymptotically stable and the controlled output satisfies H_∞ performance constraint for all nonzero exogenous disturbances under zero-initial conditions.

(3) A unified framework is given in this chapter for the case of multiple packets being used to design the H_∞ controller which can be met with in multiple radios scenario of wireless networks.

8.2 Problem formulation

Consider the following discrete-time stochastic nonlinear system:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + Dw_k + \beta_k f(x_{k-\tau_k}) \\ y_k = Cx_k \\ z_k = Lx_k \end{cases} \tag{8.1}$$

where $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^r$ is the measured output without sensor saturation, $u_k \in \mathbb{R}^p$ is the controller to be designed. $z_k \in \mathbb{R}^m$ is the controlled output; τ_k is a positive integer which denotes the time-varying delay and satisfies $0 \le \underline{\tau} \le \tau_k \le \bar{\tau}$, where $\underline{\tau}$ and $\bar{\tau}$ are known positive integers representing the minimal and maximal delays, respectively. A, B, C, D and L are known constant matrices. The Bernoulli random variable β_k is introduced to describe the random occurrence nature of nonlinearities whose statistical characteristics are:

$$\text{Prob}\{\beta_k = 1\} = E\{\beta_k\} = \bar{\beta}, \quad \text{Prob}\{\beta_k = 0\} = E\{1 - \beta_k\} = 1 - \bar{\beta},$$

For system (8.1), we give the following assumptions:

Assumption 8.1 The exogenous disturbance signal $w_k \in \mathbb{R}^q$ is energy bounded, that is, it belongs to $l_2[0, \infty)$.

Assumption 8.2 The control matrix B is of full column rank which can be satisfied for many practical systems (e.g., [25,26]).

Assumption 8.3 The nonlinear function $f(\cdot)$ satisfies the sector-bounded conditions that

$$\begin{aligned} &\text{(i) } f(0) = 0 \\ &\text{(ii) } [f(x) - f(y) - R_1(x-y)]^T[f(x) - f(y) - R_2(x-y)] \le 0 \end{aligned} \tag{8.2}$$

where $R_1 \in \mathbb{R}^{n\times n}$, $R_2 \in \mathbb{R}^{n\times n}$ are known real constant matrices, and $R_1 - R_2$ is a positive definite matrix.

Remark 8.1 The nonlinear function $f(\cdot)$ which covers the Lipschitz type of nonlinearity has broad applications: e.g., it is utilized to describe Hopfield neural networks, cellular neural networks, Chur's circuit, *etc*. [27].

The corresponding NCS schematic is shown in Figure 8.1.

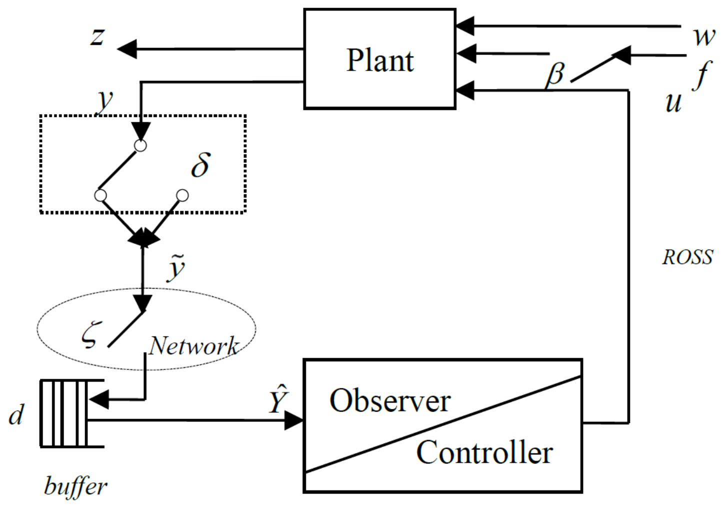

Figure 8.1 Schematic for the considered NCS.

8.2.1 Sensor saturation

In this chapter, the sensor saturations/nonlinearities which are induced by the limitation of technique and security constraints are taken into consideration. Moreover, in practical engineering, the sensor saturation may occur randomly. Thus, the randomly saturated output of the sensor is given as

$$\tilde{y}_k = \delta_k \phi(y_k) + (1 - \delta_k) y_k \tag{8.3}$$

where $\tilde{y}_k \in \mathbb{R}^r$ to be transmitted to the remote receiver through the networks is the measurements with ROSS whose random nature is described by the Bernoulli distributed random variable δ_k satisfying

$$\text{Prob}\{\delta_k = 1\} = \text{E}\{\delta_k\} = \bar{\delta},\ \text{Prob}\{\delta_k = 0\} = \text{E}\{1 - \delta_k\} = 1 - \bar{\delta},$$

Moreover, the random variable δ_k is uncorrelated with β_k.

Assumption 8.4 The saturation function $\phi(\cdot)$ belongs to $[\text{M}_1, \text{M}_2]$ with given diagonal matrices $M_1 \in \mathbb{R}^{r\times r}$, $M_2 \in \mathbb{R}^{r\times r}$, where $M_2 > M_1 \geq 0$ and satisfies the following inequality:

$$[\phi(y_k) - M_1 y_k]^T[\phi(y_k) - M_2 y_k] \leq 0$$

It is easy to see that $\phi(y_k)$ is a nonlinear function and $\phi(y_k) = sign(y_k)\min\{y_{k,max}, |y_k|\}$ is the saturated value and $y_{k,max}$ is the saturation level [28]. For technical convenience, we introduce a general nonlinear decomposition approach to deal with the sensor saturation, *i.e.*, the nonlinear function $\phi(y_k)$ can be decomposed into a linear part and a nonlinear part as

$$\phi(y_k) = \phi_s(y_k) + M_1 y_k \tag{8.4}$$

where the nonlinear part $\phi_s(y_k)$ belongs to the set Φ_s which is defined by

$$\Phi_s = \{\phi_s: \phi_s^T(y_k)(\phi_s(y_k) - M y_k) \leq 0 \tag{8.5}$$

with $M = M_2 - M_1 > 0$.

8.2.2 Network transmission

Assumption 8.5 The sampling and sending rates of the sensor and the receiving rate of the controller are synchronous and time-triggered.

Due to the limited bit rates of the communication channels, the sensor outputs will be received with delays at the receiver side during the transmission through the network links. By applying the time-stamp technique, we can know the maximum delay. If a packet is not received within the maximum delay, it can be treated as a packet dropout. Moreover, the transmission delays will lead to the packets receiving by the controller with out of sequence and there will be multiple data packets arriving at the buffer in the receiver side within one sampling period. Here, we describe the signal received by the controller $\hat{Y}_k \in \mathbb{R}^{dr}$ as follows [12]:

$$\hat{Y}_k = \begin{bmatrix} \zeta_k^{(0)}\tilde{y}_k \\ (1 - \zeta_{k-1}^{(0)})\zeta_k^{(1)}\tilde{y}_{k-1} \\ \vdots \\ \prod_{i=0}^{d-1}(1 - \zeta_{k-d+1}^{(i)})\zeta_k^{(d)}\tilde{y}_{k-d} \end{bmatrix} + \text{G}v_k \tag{8.6}$$

where $\zeta_k^{(i)} (i = 0,1, \dots, d)$ are a group of Bernoulli distributed random variables which are uncorrelated with each other and also uncorrelated with β_k and δ_k.

The probabilities of $\zeta_k^{(i)}(i = 0,1,\dots,d)$ satisfy that

$$\text{Prob}\{\zeta_k^{(i)} = 1\} = E\left\{\zeta_k^{(i)}\right\} = \bar{\zeta}_i,\ \text{Prob}\{\zeta_k^{(i)} = 0\} = E\left\{1 - \zeta_k^{(i)}\right\} = 1 - \bar{\zeta}_i$$

$v_k \in \mathbb{R}^l$, assumed to belong to $l_2[0,\infty)$, is the channel noise, G is a known constant matrix, and d is the maximal transmission delay.

Remark 8.2 Although the event-triggered scheme has been proposed to save the network bandwidth resources, some event-triggered communication schemes require additional hardware devices to continuously check the system measurements, which make difficult to put them into practice [6,29-31]. So, the traditional time-triggered scheme in which signals transfer at a fixed rate is often used for its easy implementation.

Remark 8.3 Additionally, the model in (8.6) can be met in the case of multiple radios scenario. The usage of multiple radios can significantly improve the capacity of the network by permitting an increased number of concurrent transmissions in the network [32].

To conduct conveniently for the subsequent discussion, some new variables are introduced.

Let $\alpha_{0,k} = \zeta_k^{(0)}$, $\alpha_{j,k} = \prod_{i=0}^{j-1}\left(1 - \zeta_{k-j+1}^{(i)}\right)\zeta_k^{(j)}$ $(1 \le j \le d)$, Then, we have

$$\hat{Y}_k = \begin{bmatrix} \alpha_{0,k}\tilde{y}_k \\ \alpha_{1,k}\tilde{y}_{k-1} \\ \vdots \\ \alpha_{d,k}\tilde{y}_{k-d} \end{bmatrix} + \mathrm{G}v_k$$

where the statistic characteristics of $\alpha_{i,k}(0 \le i \le d)$ are given as $E\{\alpha_{0,k}\} = \bar{\zeta}_0 = \bar{\alpha}_0$, $E\{\alpha_{j,k}\} = \prod_{i=0}^{j-1}(1 - \bar{\zeta}_i)\bar{\zeta}_j = \bar{\alpha}_j$, and $E\{\alpha_{j,k}\alpha_{l,k}\} = \bar{\alpha}_j\bar{\alpha}_l,\ j \ne l$.

Remark 8.4 From the definition of $\alpha_{i,k}(0 \le i \le d)$, we can obtain $\alpha_{i,k+i}\alpha_{j,k+j} = 0\ \ i \ne j$. As is well known, in the real network circumstance, such as the TCP/IP case, one packet is only received once. This property of networks can be described by the relation $\alpha_{i,k+i}\alpha_{j,k+j} = 0\ (i \ne j)$ which is the same constraint as in [33].

In the subsequent, the multiple packets arriving at the receiver are used to design the state observer of the system and then an observer-based controller is presented for networked stochastic nonlinear systems with transmission delays, packet dropouts as well as the randomly occurred sensor saturations.

8.2.3 Observer-based controller design

Owing to the partially known state information in real-world engineering, the state-feedback method which needs all the states to be measurable is not always satisfied in reality [34]. By taking the nonlinear disturbance of the system into consideration, we design the following dynamic controller based on a nonlinear observer for the system (8.1):

$$\begin{cases} \hat{x}_{k+1} = A_c\hat{x}_k + \bar{\beta}f\left(\hat{x}_{k-\tau_k}\right) + B_c\hat{Y}_k \\ u_k = C_c\hat{x}_k \end{cases} \tag{8.7}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the state estimation of system (8.1), and A_c, B_c, C_c are the parameters to be determined.

Remark 8.5 As for the imperfect network transmission, there are three strategies for the controller design. The first is zero-input strategy [7], the second is hold-input strategy [35], and the third is prediction-input strategy [36]. In this chapter, if we let $B_c = \begin{bmatrix} B_c^{(0)} & B_c^{(1)} & \cdots & B_c^{(d)} \end{bmatrix}$, we can obtain

$$\hat{x}_{k+1} = A_c\hat{x}_k + \sum_{i=0}^{d} \alpha_{i,k}B_c^{(i)}\tilde{y}_{k-i} + \bar{\beta}f\left(\hat{x}_{k-\tau_k}\right) + B_cGv_k$$

It can be seen that the multiple packets are used to design the controller.

8.2.4 Closed-loop system

The corresponding closed-loop system by combining the above controller in (8.7) to the original system in (8.1) and (8.4) is presented by

$$\begin{cases} \eta_{k+1} = (\bar{A} + \tilde{\bar{\xi}}_{0,k}\hat{A}^{(0)} + \tilde{\alpha}_{0,k}\breve{A}^{(0)})\eta_k + \sum_{i=1}^{d}(\bar{\xi}_i\hat{A}^{(i)} + \bar{\alpha}_i\breve{A}^{(i)})\eta_{k-i} + \sum_{i=1}^{d}(\tilde{\bar{\xi}}_{i,k}\hat{A}^{(i)} + \tilde{\alpha}_{i,k}\breve{A}^{(i)})\eta_{k-i} \\ \qquad +HF\left(\eta_{k-\tau_k}\right) + \widetilde{H}F\left(\eta_{k-\tau_k}\right) + \sum_{i=0}^{d}\bar{\xi}_i\bar{B}^{(i)}\phi_s(\bar{C}\eta_{k-i}) + \sum_{i=0}^{d}\tilde{\bar{\xi}}_{i,k}\bar{B}^{(i)}\phi_s(\bar{C}\eta_{k-i}) + \bar{D}\widetilde{w}_k \\ z_k = \bar{L}\eta_k \end{cases} \tag{8.8}$$

where

$$\eta_k = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T,\ \widetilde{w}_k = \begin{bmatrix} w_k^T & v_k^T \end{bmatrix}^T$$

$$\bar{A} = \begin{bmatrix} A & BC_c \\ \bar{\xi}_0B_c^{(0)}(M_1 - I)C + \bar{\alpha}_0B_c^{(0)}C & A_c \end{bmatrix},\ \bar{B}^{(i)} = \begin{bmatrix} 0 \\ B_c^{(i)} \end{bmatrix},\ \hat{A}^{(i)} = \begin{bmatrix} 0 & 0 \\ B_c^{(i)}(M_1 - I)C & 0 \end{bmatrix},$$

$$\breve{A}^{(i)} = \begin{bmatrix} 0 & 0 \\ B_c^{(i)}C & 0 \end{bmatrix},\ \bar{D} = \begin{bmatrix} D & 0 \\ 0 & B_cG \end{bmatrix},\ H = \begin{bmatrix} \bar{\beta} & 0 \\ 0 & \bar{\beta} \end{bmatrix},\ \widetilde{H} = \begin{bmatrix} \tilde{\beta}_k & 0 \\ 0 & 0 \end{bmatrix},$$

$$F\left(\eta_{k-\tau_k}\right) = \begin{bmatrix} f\left(x_{k-\tau_k}\right) \\ f\left(\hat{x}_{k-\tau_k}\right) \end{bmatrix},\ \bar{L} = \begin{bmatrix} L & 0 \end{bmatrix},\ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix},$$

$$\xi_{i,k} = \alpha_{i,k}\delta_{i-k},\ \bar{\xi}_i = \bar{\alpha}_i\bar{\delta},\ \tilde{\bar{\xi}}_{i,k} = \xi_{i,k} - \bar{\xi}_i,\ \tilde{\beta}_k = \beta_k - \bar{\beta},$$

$$\tilde{\alpha}_{i,k} = \alpha_{i,k} - \bar{\alpha}_i(i = 0,1,\ldots,d)$$

Remark 8.6 It is seen from (8.7) that the designed controller is suitable for the case of known nonlinear function $f(\cdot)$. If the nonlinear function is not known, we can choose the linear observer-based controller which is the special case of the nonlinear one. In this case, some variables would be redefined, that is $F(\eta_{k-\tau_k}) = \begin{bmatrix} f(\eta_{k-\tau_k}) \\ 0 \end{bmatrix}$ and $H = \begin{bmatrix} \bar{\beta} & 0 \\ 0 & 0 \end{bmatrix}$.

Up to now, the original system (8.1) is transformed to system (8.8) which is characterized by random variables $\xi_{i,k}$, $\alpha_{i,k}(i = 0,1,\dots,d)$, and β_k. Then, we will use the stochastic parameterized method to analyze and design the controller. For the subsequent derivations, we give the following statistical characteristics. From the distributions of $\xi_{i,k}$, $\alpha_{i,k}(i = 0,1,\dots,d)$, and β_k, we can easily obtain that

$$\rho_i^2 = E\{\tilde{\xi}_{i,k}^2\} = E\left\{\left(\alpha_{i,k}\delta_{k-i} - \bar{\alpha}_i\bar{\delta}\right)^2\right\} = \bar{\alpha}_i\bar{\delta}(1 - \bar{\alpha}_i\bar{\delta})$$

$$\theta_i^2 = E\{\tilde{\alpha}_{i,k}^2\} = E\left\{\left(\alpha_{i,k} - \bar{\alpha}_i\right)^2\right\} = \bar{\alpha}_i(1 - \bar{\alpha}_i)$$

$$\vartheta_i^2 = E\{\tilde{\beta}_{i,k}^2\} = E\left\{\left(\beta_k - \bar{\beta}\right)^2\right\} = \bar{\beta}(1 - \bar{\beta})$$

$$\sigma_i^2 = E\{\tilde{\xi}_{i,k}\tilde{\alpha}_{i,k}\} = E\{\left(\alpha_{i,k}\delta_{k-i} - \bar{\alpha}_i\bar{\delta}\right)\left(\alpha_{i,k} - \bar{\alpha}_i\right)\} = \bar{\alpha}_i\bar{\delta}(1 - \bar{\alpha}_i)$$

The purpose of this chapter is to develop techniques to deal with the H_∞ control problem for the discrete-time stochastic system such that, in the presence of the packet losses, delayed measurements and the RON as well as the ROSS for system (8.1), the asymptotical stability and H_∞ performance are both satisfied for the given disturbance attenuation level $\gamma > 0$. That is, the following two requirements are satisfied simultaneously:

(i) The system (8.8) is asymptotically stable in the mean square if $\tilde{w}_k = 0$, for all $\eta_0 \in \mathbb{R}^{2n}$, *i.e.*, the following equation holds:

$$\lim_{k\to\infty} E\{||\eta_k||^2\} = 0$$

(ii) Under zero initial conditions, the controlled output z_k satisfies the following H_∞ constraint:

$$\sum_{k=0}^{\infty} E\{\|z_k\|^2\} < \gamma^2 \sum_{k=0}^{\infty} E\{\|\tilde{w}_k\|^2\} \tag{8.9}$$

for all nonzero $\tilde{w}_k$, and $\gamma > 0$ is a prescribed scalar.

8.3 Main results

The following Theorem provides a sufficient condition for the closed-loop system (8.8) to be asymptotically mean-square stable and also for the controlled output z_k to satisfy the H_∞ disturbance attenuation requirement.

Lemma 8.1 [37] For matrices A, $P_0 > 0$ and $P_1 > 0$, the following inequality:

$$A^T P_1 A - P_0 < 0$$

is equivalent to that there exists a matrix W such that

$$\begin{bmatrix} -P_0 & * \\ WA & P_1 - W - W^T \end{bmatrix} < 0$$

Theorem 8.1 Consider the discrete-time network-based system (8.1) and suppose that the controller parameters A_c, B_c and C_c are given. Then, the closed-loop system (8.8) is asymptotically stable in the mean square, and when $\widetilde{w}_k \neq 0$, the H_∞ norm constraint (8.9) is achieved for a given scalar $\gamma > 0$ if there exist matrices $P > 0$, $Q_l > 0\ (l = 1,2, \dots d)$, $S > 0$ and positive scalars $\lambda_1 > 0$ and $\lambda_2 > 0$, such that the following matrix inequality holds

$$\begin{bmatrix} \Xi_{11} & * & * & * \\ \Xi_{21} & \Xi_{22} & * & * \\ \Xi_{31} & 0 & \Xi_{33} & * \\ \Xi_{41} & 0 & 0 & \Xi_{44} \end{bmatrix} < 0 \tag{8.10}$$

where

$$\Xi_{11} = \begin{bmatrix} \sum_{j=1}^{d} Q_j - P - \lambda_1 I \otimes \vec{R}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\bar{Q} & 0 & 0 & 0 & 0 \\ \lambda_1 I \otimes \vec{R}_2 & 0 & (\bar{\tau} - \underline{\tau} + 1)S - \lambda_1 I & 0 & 0 & 0 \\ 0 & 0 & 0 & -S & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_2 I_{d+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$\Xi_{21} = \begin{bmatrix} P\bar{A} & P(\bar{\Lambda}_1 + \bar{\Lambda}_2) & 0 & PH & P\bar{\Lambda}_3 & P\bar{D} \\ \rho_0 P\hat{A}^{(0)} & 0 & 0 & 0 & P\bar{\Lambda}_4 & 0 \\ 0 & (I_d \otimes P)\widetilde{\Lambda}_1 & 0 & 0 & (I_d \otimes P)\widetilde{\Lambda}_3 & 0 \\ \upsilon_0 P\check{A}^{(0)} & 0 & 0 & 0 & 0 & 0 \\ 0 & (I_d \otimes P)\widetilde{\Lambda}_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Xi_{31} = \begin{bmatrix} \sigma_0 P\hat{A}^{(0)} & 0 & 0 & 0 & 0 & 0 \\ 0 & (I_d \otimes P)\widetilde{\Lambda}_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (I_{d+1} \otimes P)\widetilde{\Lambda}_5 & 0 \\ 0 & 0 & 0 & \vartheta P\bar{H} & 0 & 0 \end{bmatrix}$$

$$\Xi_{41} = \begin{bmatrix} \bar{L} & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 M & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 (I_d \otimes M) & 0 & 0 & 0 & 0 \end{bmatrix}, \ \bar{H} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Xi_{22} = diag\{-P \quad -P \quad -(I_d \otimes P) \quad -P \quad -(I_d \otimes P)\}$$

$$\Xi_{33} = diag\{-P \quad -(I_d \otimes P) \quad -(I_d \otimes P) \quad -P\}$$

$$\Xi_{44} = diag\{-I \quad -I \quad -I_d\}, \ \bar{Q} = diag\{Q_1 \quad Q_2 \quad \cdots \quad Q_d\}$$

$$\bar{\Lambda}_1 = [\bar{\xi}_1 \hat{A}^{(1)} \quad \bar{\xi}_2 \hat{A}^{(2)} \quad \cdots \quad \bar{\xi}_d \hat{A}^{(d)}]$$

$$\bar{\Lambda}_2 = [\bar{\alpha}_1 \check{A}^{(1)} \quad \bar{\alpha}_2 \check{A}^{(2)} \quad \cdots \quad \bar{\alpha}_d \check{A}^{(d)}]$$

$$\bar{\Lambda}_3 = [\bar{\xi}_0 \bar{B}^{(0)} \quad \bar{\xi}_1 \bar{B}^{(1)} \quad \cdots \quad \bar{\xi}_d \bar{B}^{(d)}]$$

$$\bar{\Lambda}_4 = [\rho_0 \bar{B}^{(0)} \quad 0 \quad \cdots \quad 0]$$

$$\widetilde{\Lambda}_1 = diag\{\rho_1 \hat{A}^{(1)} \quad \rho_2 \hat{A}^{(2)} \quad \cdots \quad \rho_d \hat{A}^{(d)}\}$$
$$\widetilde{\Lambda}_2 = diag\{\upsilon_1 \breve{A}^{(1)} \quad \upsilon_2 \breve{A}^{(2)} \quad \cdots \quad \upsilon_d \breve{A}^{(d)}\}$$
$$\widetilde{\Lambda}_3 = [0 \quad diag\{\rho_1 \bar{B}^{(1)} \quad \rho_2 \bar{B}^{(2)} \quad \cdots \quad \rho_d \bar{B}^{(d)}\}]$$
$$\widetilde{\Lambda}_4 = diag\{\sigma_1 \hat{A}^{(1)} \quad \sigma_2 \hat{A}^{(2)} \quad \cdots \quad \sigma_d \hat{A}^{(d)}\}$$
$$\widetilde{\Lambda}_5 = diag\{\sigma_0 \bar{B}^{(0)} \quad \sigma_1 \bar{B}^{(1)} \quad \cdots \quad \sigma_d \bar{B}^{(d)}\}$$
$$\widetilde{\Lambda}_6 = diag\{\rho_0 \bar{B}^{(0)} \quad \rho_1 \bar{B}^{(1)} \quad \cdots \quad \rho_d \bar{B}^{(d)}\}$$
$$\vec{R}_1 = (R_1^T R_2 + R_2^T R_1)/2,\ \vec{R}_2 = -(R_1^T + R_2^T)/2,$$
$$\upsilon_l^2 = 2\sigma_l^2 + \theta_l^2 (l = 0,1,\cdots,d).$$

Proof Choose the Lyapunov functional as follows

$$V_k = \sum_{i=1}^{4} V_{i,k} \tag{8.11}$$

where

$$V_{1,k} = \eta_k^T P \eta_k \tag{8.12}$$

$$V_{2,k} = \sum_{j=1}^{d} \sum_{i=k-j}^{k-1} \eta_i^T Q_j \eta_i \tag{8.13}$$

$$V_{3,k} = \sum_{i=k-\tau_k}^{k-1} F^T(\eta_i) S F(\eta_i) \tag{8.14}$$

$$V_{4,k} = \sum_{j=k-\bar{\tau}+1}^{k-\underline{\tau}} \sum_{i=j}^{k-1} F^T(\eta_i) S F(\eta_i) \tag{8.15}$$

Then, we calculate the difference of the Lyapunov function (8.11) along the trajectory of (8.8). By substituting (8.8) into $\mathrm{E}\{\Delta V_{1k}\} = \mathrm{E}\{\eta_{k+1}^T P \eta_{k+1} - \eta_k^T P \eta_k\}$, we can obtain

$$\begin{aligned}
E\{\Delta V_{1k}\} = {} & \eta_k^T \Omega_{\eta\eta}^{(0)} \eta_k + 2\sum_{i=1}^{d} \eta_k^T \Omega_{\eta\eta}^{(i)} \eta_{k-i} + 2\eta_k^T \Omega_{\eta F} F(\eta_{k-\tau_k}) + 2\eta_k^T \Omega_{\eta\phi}^{(0)} \phi_s(\bar{C}\eta_k) \\
& + 2\sum_{i=0}^{d} \eta_k^T \Omega_{\eta\phi}^{(i)} \phi_s(\bar{C}\eta_k) + \sum_{i=1}^{d} \eta_{k-i}^T \Omega_{\eta\eta}^{(ii)} \eta_{k-i} + 2\sum_{i=1}^{d}\sum_{j=1}^{d} \eta_{k-i}^T \Omega_{\eta\eta}^{(ij)} \eta_{k-j} \\
& + 2\sum_{i=1}^{d} \eta_{k-i}^T \Omega_{\eta F}^{(ii)} F(\eta_{k-\tau_k}) + 2\sum_{i=0}^{d} \eta_{k-i}^T \Omega_{\eta\phi}^{(ii)} \phi_s(\bar{C}\eta_{k-i}) \\
& + 2\sum_{i=1}^{d}\sum_{j=1}^{d} \eta_{k-i}^T \Omega_{\eta\eta}^{(ij)} \eta_{k-j} + 2\sum_{i=1}^{d}\sum_{j=0}^{d} \eta_{k-i}^T \Omega_{\eta\phi}^{(ij)} \phi_s(\bar{C}\eta_{k-i}) \\
& + F^T(\eta_{k-\tau_k}) \Omega_{FF} F(\eta_{k-\tau_k}) + 2\sum_{i=0}^{d} F^T(\eta_{k-\tau_k}) \Omega_{F\phi}^{(i)} \phi_s(\bar{C}\eta_{k-i}) \\
& + \sum_{i=0}^{d} \phi_s^T(\bar{C}\eta_{k-i}) \Omega_{\phi\phi}^{(ii)} \phi_s(\bar{C}\eta_{k-i}) + 2\sum_{i=0}^{d}\sum_{j=0}^{d} \phi_s^T(\bar{C}\eta_{k-i}) \Omega_{\phi\phi}^{(ij)} \phi_s(\bar{C}\eta_{k-j})
\end{aligned} \tag{8.16}$$

$$+2\eta_k^T\bar{A}^T P\bar{D}\widetilde{w}_k + 2\sum_{i=1}^{d}\bar{\xi}_i\,\eta_{k-i}^T\hat{A}^{(i)T}P\bar{D}\widetilde{w}_k + 2\sum_{i=1}^{d}\bar{\alpha}_i\eta_{k-i}^T\check{A}^{(i)T}P\bar{D}\widetilde{w}_k + 2\,F^T(\eta_{k-\tau_k})H^T P\bar{D}\widetilde{w}_k$$

$$+\,2\sum_{i=0}^{d}\bar{\xi}_i\phi_s^T(\bar{C}\eta_{k-i})\bar{B}^{(i)T}P\bar{D}\widetilde{w}_k + \widetilde{w}_k^T\bar{D}^T P\bar{D}\widetilde{w}_k$$

where

$$\Omega_{\eta\eta}^{(0)} = \bar{A}^T P\bar{A} + \rho_0^2\hat{A}^{(0)T}P\hat{A}^{(0)} + \theta_0^2\check{A}^{(0)T}P\check{A}^{(0)} + 2\sigma_0^2\hat{A}^{(0)T}P\check{A}^{(0)} - P,$$

$$\Omega_{\eta\eta}^{(i)} = \bar{\xi}_i\bar{A}^T P\hat{A}^{(i)} + \alpha_i\bar{A}^T P\check{A}^{(i)},$$

$$\Omega_{\eta F} = \bar{A}^T PH,$$

$$\Omega_{\eta\phi}^{(0)} = \rho_0^2\hat{A}^{(0)T}P\bar{B}^{(0)} + \sigma_0^2\check{A}^{(0)T}P\bar{B}^{(0)},\ \Omega_{\eta\phi}^{(i)} = \bar{\xi}_i\bar{A}^T P\bar{B}^{(i)}$$

$$\Omega_{\eta\eta}^{(ii)} = \rho_i^2\hat{A}^{(i)T}P\hat{A}^{(i)} + \theta_i^2\check{A}^{(i)T}P\check{A}^{(i)} + 2\sigma_i^2\hat{A}^{(i)T}P\check{A}^{(i)},$$

$$\Omega_{\eta\eta}^{(ij)} = \bar{\xi}_i\bar{\xi}_j\hat{A}^{(i)T}P\hat{A}^{(j)} + \bar{\alpha}_i\bar{\alpha}_j\check{A}^{(i)T}P\check{A}^{(j)} + \bar{\xi}_i\bar{\alpha}_j\hat{A}^{(i)T}P\check{A}^{(j)},$$

$$\Omega_{\eta F}^{(i)} = \bar{\xi}_i\hat{A}^{(i)T}PH + \bar{\alpha}_i\check{A}^{(i)T}PH,$$

$$\Omega_{\eta\phi}^{(i)} = \rho_i^2\hat{A}^{(i)T}P\bar{B}^{(i)} + \sigma_i^2\check{A}^{(i)T}P\bar{B}^{(i)},$$

$$\Omega_{\eta\phi}^{(ij)} = \bar{\xi}_i\bar{\xi}_j\hat{A}^{(i)T}P\bar{B}^{(j)} + \bar{\alpha}_i\bar{\xi}_j\check{A}^{(i)T}P\bar{B}^{(j)},$$

$$\Omega_{FF} = H^T PH + \widetilde{H}^T P\widetilde{H},\ \Omega_{F\phi}^{(i)} = \bar{\xi}_i H^T P\bar{B}^{(i)},$$

$$\Omega_{\phi\phi}^{(ii)} = \rho_i^2\bar{B}^{(i)T}\,P\bar{B}^{(i)},\ \Omega_{\phi\phi}^{(ij)} = \bar{\xi}_i\bar{\xi}_j\bar{B}^{(i)T}\,P\bar{B}^{(j)}$$

Next, it can be derived that

$$E\{\Delta V_{2k}\} = \sum_{j=1}^{d}\left(\eta_k^T Q_j\eta_k - \eta_{k-j}^T Q_j\eta_{k-j}\right) \tag{8.17}$$

$$E\{\Delta V_{3k}\} \le E\left\{F^T(\eta_k)SF(\eta_k) - F^T(\eta_{k-\tau_k})SF(\eta_{k-\tau_k}) + \sum_{i=k-\bar{\tau}+1}^{k-\underline{\tau}}F^T(\eta_i)SF(\eta_i)\right\} \tag{8.18}$$

$$E\{\Delta V_{4k}\} \le E\{(\bar{\tau}-\underline{\tau})F^T(\eta_k)SF(\eta_k)\} - E\left\{\sum_{i=k-\bar{\tau}+1}^{k-\underline{\tau}}F^T(\eta_i)SF(\eta_i)\right\} \tag{8.19}$$

For the corresponding solution η_k of the system (8.8), we denote

$$\tilde{\eta}_k = [\eta_{k-1}^T \quad \eta_{k-2}^T \quad \cdots \quad \eta_{k-d}^T]^T,\ \bar{\eta}_k = [\eta_k^T \quad \tilde{\eta}_k^T]^T,$$

$$\widetilde{\Phi}_k = [\phi_s^T(\bar{C}\eta_k) \quad \phi_s^T(\bar{C}\eta_{k-1}) \quad \cdots \quad \phi_s^T(\bar{C}\eta_{k-d})]^T,$$

$$\tilde{\varsigma}_k = \left[\eta_k^T \quad \tilde{\eta}_k^T \quad F^T(\eta_k) \quad F^T(\eta_{k-\tau_k}) \quad \widetilde{\Phi}_k^T\right]^T,$$

$$\tilde{\varsigma}_k = \left[\eta_k^T \quad \tilde{\eta}_k^T \quad F^T(\eta_k) \quad F^T(\eta_{k-\tau_k}) \quad \widetilde{\Phi}_k^T \quad \widetilde{w}_k^T\right]^T,$$

In the following, we first prove the asymptotical stability of the closed-loop system in (8.8) with $\widetilde{w}_k = 0$. Considering (8.16), (8.17), (8.18) and (8.19), and from the elementary inequality $2a^T b \le a^T a + b^T b$, we have

$$2\sum_{i=1}^{d}\sigma_i^2\eta_{k-i}^T\hat{A}^{(i)T}P\breve{A}^{(i)}\eta_{k-i}\le\sum_{i=1}^{d}\sigma_i^2\eta_{k-i}^T\hat{A}^{(i)T}P\hat{A}^{(i)}\eta_{k-i}+\sum_{i=1}^{d}\sigma_i^2\eta_{k-i}^T\breve{A}^{(i)T}P\breve{A}^{(i)}\eta_{k-i} \tag{8.20}$$

$$\begin{aligned}&2\sum_{i=1}^{d}\sigma_i^2\eta_{k-i}^T\breve{A}^{(i)T}P\bar{B}^{(i)}\phi_s(\bar{C}\eta_{k-i})\\ &\le\sum_{i=1}^{d}\sigma_i^2\eta_{k-i}^T\breve{A}^{(i)T}P\breve{A}^{(i)}\eta_{k-i}+\sum_{i=1}^{d}\sigma_i^2\phi_s^T(\bar{C}\eta_{k-i})\bar{B}^{(i)T}P\bar{B}^{(i)}\phi_s(\bar{C}\eta_{k-i})\end{aligned} \tag{8.21}$$

Furthermore, it follows from (8.2) that

$$[F(\eta_k)-(I\otimes R_1)\eta_k]^T[F(\eta_k)-(I\otimes R_2)\eta_k]\le 0 \tag{8.22}$$

Then, we can obtain from (8.16)-(8.21) combined with the constraints (8.22) and (8.5) that

$$\begin{aligned}E\{\Delta V_k\}&\le E\left\{\sum_{i=1}^{4}\Delta V_i-\lambda_1[F(\eta_k)-(I\otimes R_1)\eta_k]^T[F(\eta_k)-(I\otimes R_2)\eta_k]-2\lambda_2[\widetilde{\Phi}_k^T\widetilde{\Phi}_k\right.\\ &\qquad\left.-\widetilde{\Phi}_k^T(I_{d+1}\otimes M)\bar{\eta}_k]\right\}\\ &\le E\left\{\sum_{i=1}^{4}\Delta V_i-\lambda_1[F(\eta_k)-(I\otimes R_1)\eta_k]^T[F(\eta_k)-(I\otimes R_2)\eta_k]-\lambda_2\widetilde{\Phi}_k^T\widetilde{\Phi}_k\right.\\ &\qquad\left.+\lambda_2\eta_k^T(I_{d+1}\otimes M^TM)\bar{\eta}_k]\right\}\le\tilde{\varsigma}_k^T\bar{\Pi}\tilde{\varsigma}_k\end{aligned} \tag{8.23}$$

where

$$\bar{\Pi}=\begin{bmatrix}\bar{\Pi}_{11} & * & * & * & *\\ \bar{\Pi}_{21} & \bar{\Pi}_{22} & * & * & *\\ \bar{\Pi}_{31} & 0 & \bar{\Pi}_{33} & * & *\\ \bar{\Pi}_{41} & \bar{\Pi}_{42} & 0 & \bar{\Pi}_{44} & *\\ \bar{\Pi}_{51} & \bar{\Pi}_{52} & 0 & \bar{\Pi}_{54} & \bar{\Pi}_{55}\end{bmatrix}$$

$$\bar{\Pi}_{11}=\bar{A}^TP\bar{A}+(\rho_0^2+\sigma_0^2)\hat{A}^{(0)T}P\hat{A}^{(0)}+v_0^2\breve{A}^{(0)T}P\breve{A}^{(0)}-P+\sum\nolimits_{j=1}^{d}Q_j-\lambda_1I\otimes\vec{R}_1+\lambda_2\,M^TM$$

$$\bar{\Pi}_{21}=[\bar{\xi}_1\bar{A}^TP\hat{A}^{(1)}+\alpha_1\bar{A}^TP\breve{A}^{(1)}\quad\cdots\quad\bar{\xi}_d\bar{A}^TP\hat{A}^{(d)}+\alpha_d\bar{A}^TP\breve{A}^{(d)}]^T$$

$$\bar{\Pi}_{31}=\lambda_1I\otimes\vec{R}_2,\ \bar{\Pi}_{41}=H^TP\bar{A}$$

$$\bar{\Pi}_{51}=[\bar{\xi}_0\bar{A}^TP\bar{B}^{(0)}+\rho_0^2\hat{A}^{(0)T}P\bar{B}^{(0)}\quad\bar{\xi}_1\bar{A}^TP\bar{B}^{(1)}\quad\cdots\quad\bar{\xi}_d\bar{A}^TP\bar{B}^{(d)}]^T$$

$$\bar{\Pi}_{22}=(\bar{\Lambda}_1+\bar{\Lambda}_2)^TP(\bar{\Lambda}_1+\bar{\Lambda}_2)+\widetilde{\Lambda}_1^TP_d\widetilde{\Lambda}_1+\widetilde{\Lambda}_2^TP_d\widetilde{\Lambda}_2+\widetilde{\Lambda}_4^TP_d\widetilde{\Lambda}_4+\lambda_2(I_d\otimes M^TM)-\bar{Q}$$

$$\bar{\Pi}_{42}=[\bar{\xi}_1H^TP\hat{A}^{(1)}+\alpha_1H^TP\breve{A}^{(1)}\quad\cdots\quad\bar{\xi}_dH^TP\hat{A}^{(d)}+\alpha_dH^TP\breve{A}^{(d)}]$$

$$\bar{\Pi}_{52}=(\bar{\Lambda}_1+\bar{\Lambda}_2)^TP\bar{\Lambda}_3+\widetilde{\Lambda}_1^TP_d\widetilde{\Lambda}_3$$

$$\bar{\Pi}_{33}=(\bar{\tau}-\underline{\tau}+1)S-\lambda_1I,\ \bar{\Pi}_{44}=H^TPH+\widetilde{H}^TP\widetilde{H}-S$$

$$\bar{\Pi}_{54}=[\bar{\xi}_0H^TP\bar{B}^{(0)}\quad\bar{\xi}_1H^TP\bar{B}^{(1)}\quad\cdots\quad\bar{\xi}_dH^TP\bar{B}^{(d)}]^T$$

$$\bar{\Pi}_{55}=\bar{\Lambda}_3^TP\bar{\Lambda}_3+\widetilde{\Lambda}_6^TP_{d+1}\widetilde{\Lambda}_6+\widetilde{\Lambda}_5^TP_{d+1}\widetilde{\Lambda}_5-\lambda_2I_{d+1}$$

By using Schur complement, we have the conclusion that (8.10) implies $\bar{\Pi}<0$. Therefore, $E\{\Delta V_k\}<0$ and then the closed-loop system (8.8) with $\widetilde{w}_k=0$ is asymptotically mean-square stable.

In order to analyze the H_∞ performance of the system (8.8), we introduce the following index:

$$J(n)=\sum_{k=0}^{n}E\{z_k^T z_k-\gamma^2\widetilde{w}_k^T\widetilde{w}_k\}$$

Obviously, our aim is to show $J(n)<0$ under the zero-initial conditions. It follows from (8.7) and (8.23) that

$$\begin{aligned} J(n)&=\sum_{k=0}^{n}(E\{z_k^T z_k\}-\gamma^2E\{\widetilde{w}_k^T\widetilde{w}_k\}+E\{\Delta V_k\}-E\{V_{n+1}\})\\ &\le\sum_{k=0}^{n}(E\{z_k^T z_k\}-\gamma^2E\{\widetilde{w}_k^T\widetilde{w}_k\}+E\{\Delta V_k\})\le\sum_{k=0}^{n}E\{\varsigma_k^T\Pi\varsigma_k\} \end{aligned}\tag{8.24}$$

where

$$\Pi=\begin{bmatrix}\overline{\Pi}_{11}+\overline{L}^T\overline{L} & * & * & * & * & *\\ \overline{\Pi}_{21} & \overline{\Pi}_{22} & * & * & * & *\\ \overline{\Pi}_{31} & 0 & \overline{\Pi}_{33} & * & * & *\\ \overline{\Pi}_{41} & \overline{\Pi}_{42} & 0 & \overline{\Pi}_{44} & * & *\\ \overline{\Pi}_{51} & \overline{\Pi}_{52} & 0 & \overline{\Pi}_{54} & \overline{\Pi}_{55} & *\\ \Pi_{61} & \Pi_{62} & 0 & \Pi_{64} & \Pi_{65} & \Pi_{66}\end{bmatrix}$$

$$\Pi_{61}=\overline{D}^TP\bar{A},\ \Pi_{64}=\overline{D}^TPH,\ \Pi_{66}=\overline{D}^TP\overline{D}-\gamma^2I$$

$$\Pi_{62}=[\bar{\xi}_1\overline{D}^TP\hat{A}+\alpha_1\overline{D}^TP\breve{A}\quad\cdots\quad\bar{\xi}_d\overline{D}^TP\hat{A}+\alpha_d\overline{D}^TP\breve{A}]^T$$

$$\Pi_{65}=[\bar{\xi}_0\overline{D}^TP\bar{B}\quad\bar{\xi}_1\overline{D}^TP\bar{B}\quad\cdots\quad\bar{\xi}_d\overline{D}^TP\bar{B}]^T$$

By using the Schur Complement Lemma, it can be concluded from (8.10) that $J(n)<0$. Since the system (8.8) is asymptotically mean-square stable by letting $n\to\infty$, it is clear to see that the inequality $\sum_{k=0}^{\infty}E\{\|z_k\|^2\}<\gamma^2\sum_{k=0}^{\infty}E\{\|\widetilde{w}_k\|^2\}$ holds under the zero initial condition, which completes the proof of Theorem 8.1.

Remark 8.7 To conduct the random and bounded transmission delays, another popular method is the delay fractioning method where the lower bound of delay d_m can always be described by $d_m=\tau m$ with τ and m being positive integers [38]. In our derivation, it can be seen as a special case of the delay partitioning method where the τ and m are both equal to 1. Another method called as the average dwell time (ADT) scheme is usually used in switch complex networks for asynchronous state estimation which means the average time of each subsystem activating should be no less than a fixed time interval, and this average activating time is defined as the ADT.

Having established the analysis result for the addressed control problem, we are now ready to discuss the corresponding controller design issue in the following theorem.

Theorem 8.2 For the discrete-time stochastic system (8.1) with possible consecutive packet dropouts, delayed measurements and randomly varying nonlinearities, there exists a controller in the form of (8.7) such that the closed-loop system (8.8) is asymptotically stable in the mean square under $\tilde{w}_k = 0$ and also satisfies the condition (8.9) under zero initial conditions for a prescribed H_∞ performance index $\gamma > 0$ if there exist matrices $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, $Q_{j1} > 0$, $Q_{j2} > 0$, $Q_{j3} > 0$ $(j = 1,2,\cdots,d)$, $S_1 > 0$, $S_2 > 0$, $S_3 > 0$, $\tilde{A}_c$, $\tilde{B}_c$, $\tilde{C}_c$, $W_1 = \begin{bmatrix} W_{11} & W_{12} \\ \varepsilon_1 U_{(n-p)\times p} W_{11} & W_{22} \end{bmatrix} T$, W_2, and $W_3 = \begin{bmatrix} \varepsilon_2 W_{11} & W_{32} \\ \varepsilon_3 U_{(n-p)\times p} W_{11} & W_{42} \end{bmatrix} T$ such that the following LMI holds:

$$\begin{bmatrix} \Xi_{11} & * & * & * \\ \bar{\Psi}_{21} & \bar{\Psi}_{22} & * & * \\ \bar{\Psi}_{31} & 0 & \bar{\Psi}_{33} & * \\ \Xi_{41} & 0 & 0 & \Xi_{44} \end{bmatrix} < 0 \tag{8.25}$$

where

$$P_1 = \begin{bmatrix} P_1 & * \\ P_2 & P_3 \end{bmatrix},\ S = \begin{bmatrix} S_1 & * \\ S_2 & S_3 \end{bmatrix},\ Q_j = \begin{bmatrix} Q_{j1} & * \\ Q_{j2} & Q_{j3} \end{bmatrix},\ j = 1,2,\cdots,d$$

$$\bar{\Psi}_{21} = \begin{bmatrix} \bar{\Psi}_{211} & \bar{\Psi}_{212} & 0 & \bar{\Psi}_{214} & \bar{\Psi}_{215} & \bar{\Psi}_{216} \\ \bar{\Psi}_{221} & 0 & 0 & 0 & \bar{\Psi}_{225} & 0 \\ 0 & \bar{\Psi}_{232} & 0 & 0 & \bar{\Psi}_{235} & 0 \\ \bar{\Psi}_{241} & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\Psi}_{252} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\Psi}_{211} = \begin{bmatrix} W_1 A + \bar{\xi}_0 \tilde{B}_c^{(0)}(M_1 - I)C + \bar{\alpha}_0 \tilde{B}_c^{(0)} C & \Sigma_1 \tilde{C}_c + \tilde{A}_c \\ W_3 A + \bar{\xi}_0 \tilde{B}_c^{(0)}(M_1 - I)C + \bar{\alpha}_0 \tilde{B}_c^{(0)} C & \Sigma_3 \tilde{C}_c + \tilde{A}_c \end{bmatrix}$$

$$\bar{\Psi}_{212} = \begin{bmatrix} \bar{\xi}_1 \tilde{B}_c^{(1)}(M_1 - I)C & 0 & \cdots & \bar{\xi}_d \tilde{B}_c^{(d)}(M_1 - I)C & 0 \\ \bar{\xi}_1 \tilde{B}_c^{(1)}(M_1 - I)C & 0 & \cdots & \bar{\xi}_d \tilde{B}_c^{(d)}(M_1 - I)C & 0 \end{bmatrix} + \begin{bmatrix} \bar{\alpha}_1 \tilde{B}_c^{(1)} C & 0 & \cdots & \bar{\alpha}_d \tilde{B}_c^{(d)} C & 0 \\ \bar{\alpha}_1 \tilde{B}_c^{(1)} C & 0 & \cdots & \bar{\alpha}_d \tilde{B}_c^{(d)} C & 0 \end{bmatrix}$$

$$\bar{\Psi}_{214} = \begin{bmatrix} \bar{\beta} W_1 & \bar{\beta} W_2 \\ \bar{\beta} W_3 & \bar{\beta} W_2 \end{bmatrix},\ \bar{\Psi}_{216} = \begin{bmatrix} W_1 D & \tilde{B}_c G \\ W_3 D & \tilde{B}_c G \end{bmatrix},\ \bar{\Psi}_{215} = \begin{bmatrix} \bar{\xi}_0 \tilde{B}_c^{(0)} & \bar{\xi}_1 \tilde{B}_c^{(1)} & \cdots & \bar{\xi}_d \tilde{B}_c^{(d)} \\ \bar{\xi}_0 \tilde{B}_c^{(0)} & \bar{\xi}_1 \tilde{B}_c^{(1)} & \cdots & \bar{\xi}_d \tilde{B}_c^{(d)} \end{bmatrix}$$

$$\bar{\Psi}_{221} = \begin{bmatrix} \rho_0 \tilde{B}_c^{(0)}(M_1 - I)C & 0 \\ \rho_0 \tilde{B}_c^{(0)}(M_1 - I)C & 0 \end{bmatrix},\ \bar{\Psi}_{225} = \begin{bmatrix} \rho_0 \tilde{B}_c^{(0)} & 0 & \cdots & 0 \\ \rho_0 \tilde{B}_c^{(0)} & 0 & \cdots & 0 \end{bmatrix}$$

$$\bar{\Psi}_{232} = diag\left\{ \begin{bmatrix} \rho_1 \tilde{B}_c^{(1)}(M_1 - I)C & 0 \\ \rho_1 \tilde{B}_c^{(1)}(M_1 - I)C & 0 \end{bmatrix} \ \cdots \ \begin{bmatrix} \rho_d \tilde{B}_c^{(d)}(M_1 - I)C & 0 \\ \rho_d \tilde{B}_c^{(d)}(M_1 - I)C & 0 \end{bmatrix} \right\}$$

$$\bar{\Psi}_{235} = \left[0 \quad diag\left\{ \begin{bmatrix} \rho_1 \tilde{B}_c^{(1)} \\ \rho_1 \tilde{B}_c^{(1)} \end{bmatrix} \ \cdots \ \begin{bmatrix} \rho_d \tilde{B}_c^{(d)} \\ \rho_d \tilde{B}_c^{(d)} \end{bmatrix} \right\} \right]$$

$$\bar{\Psi}_{241} = \begin{bmatrix} \upsilon_0 \tilde{B}_c^{(0)} C & 0 \\ \upsilon_0 \tilde{B}_c^{(0)} C & 0 \end{bmatrix},\ \bar{\Psi}_{252} = diag\left\{ \begin{bmatrix} \upsilon_1 \tilde{B}_c^{(1)} C & 0 \\ \upsilon_1 \tilde{B}_c^{(1)} C & 0 \end{bmatrix} \ \cdots \ \begin{bmatrix} \upsilon_d \tilde{B}_c^{(d)} C & 0 \\ \upsilon_d \tilde{B}_c^{(d)} C & 0 \end{bmatrix} \right\}$$

$$\bar{\Psi}_{31} = \begin{bmatrix} \bar{\Psi}_{311} & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\Psi}_{322} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\Psi}_{335} & 0 \\ 0 & 0 & 0 & \bar{\Psi}_{344} & 0 & 0 \end{bmatrix}$$

$$\bar{\Psi}_{311} = \begin{bmatrix} \sigma_0 \tilde{B}_c^{(0)}(M_1 - I)C & 0 \\ \sigma_0 \tilde{B}_c^{(0)}(M_1 - I)C & 0 \end{bmatrix},$$

$$\bar{\Psi}_{322} = diag\left\{\begin{bmatrix} \sigma_1 \tilde{B}_c^{(1)}(M_1 - I)C & 0 \\ \sigma_1 \tilde{B}_c^{(1)}(M_1 - I)C & 0 \end{bmatrix} \cdots \begin{bmatrix} \sigma_d \tilde{B}_c^{(d)}(M_1 - I)C & 0 \\ \sigma_d \tilde{B}_c^{(d)}(M_1 - I)C & 0 \end{bmatrix}\right\}$$

$$\bar{\Psi}_{235} = diag\left\{\begin{bmatrix} \sigma_0 \tilde{B}_c^{(1)} \\ \sigma_0 \tilde{B}_c^{(1)} \end{bmatrix} \begin{bmatrix} \sigma_1 \tilde{B}_c^{(1)} \\ \sigma_1 \tilde{B}_c^{(1)} \end{bmatrix} \cdots \begin{bmatrix} \sigma_d \tilde{B}_c^{(d)} \\ \sigma_d \tilde{B}_c^{(d)} \end{bmatrix}\right\}, \ \bar{\Psi}_{344} = \begin{bmatrix} \vartheta W_1 & 0 \\ \vartheta W_3 & 0 \end{bmatrix},$$

$$\hat{P} = \begin{bmatrix} P_1 - W_1 - W_1^T & * \\ P_2 - W_3 - W_2^T & P_3 - W_2 - W_2^T \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} I \\ \varepsilon_1 U_{(n-p)\times p} I \end{bmatrix}, \ \Sigma_3 = \begin{bmatrix} \varepsilon_2 I \\ \varepsilon_3 U_{(n-p)\times p} I \end{bmatrix}$$

$$\bar{\Psi}_{22} = diag\{\hat{P} \quad \hat{P} \quad I_d \otimes \hat{P} \quad \hat{P} \quad I_d \otimes \hat{P}\},$$

$$\bar{\Psi}_{33} = diag\{\hat{P} \quad I_d \otimes \hat{P} \quad I_d \otimes \hat{P} \quad \hat{P}\}.$$

And other parameters are defined as in Theorem 8.1. Furthermore, the controller parameters can be designed as

$$A_c = W_2^{-1}\tilde{A}_c, \ B_c = W_2^{-1}\tilde{B}_c, \ C_c = W_{11}^{-1}\tilde{C}_c \tag{8.26}$$

Proof From Theorem 8.1, by using Lemma 8.1, we have

$$\begin{bmatrix} \Xi_{11} & * & * & * \\ \Psi_{21} & \bar{\Psi}_{22} & * & * \\ \Psi_{31} & 0 & \bar{\Psi}_{33} & * \\ \Xi_{41} & 0 & 0 & \Xi_{44} \end{bmatrix} < 0 \tag{8.27}$$

where Ψ_{21} and Ψ_{31} are the matrices of Ξ_{21} and Ξ_{31} in Theorem 8.1 with P replaced by W. Now, let us partition W as $W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_2 \end{bmatrix}$ where W_1 and W_2 are all nonsingular without loss of generality. Furthermore, partition P and Q as $P = \begin{bmatrix} P_1 & * \\ P_2 & P_3 \end{bmatrix}$, $Q_j = \begin{bmatrix} Q_{j1} & * \\ Q_{j2} & Q_{j3} \end{bmatrix}, j = 1,2,\cdots,d$, we can obtain that

$$\begin{bmatrix} \Xi_{11} & * & * & * \\ \bar{\Theta}_{21} & \bar{\Theta}_{22} & * & * \\ \bar{\Theta}_{31} & 0 & \bar{\Theta}_{33} & * \\ \Xi_{41} & 0 & 0 & \Xi_{44} \end{bmatrix} < 0 \tag{8.28}$$

where

$$\bar{\Theta}_{21} = \begin{bmatrix} \bar{\Theta}_{211} & \bar{\Theta}_{212} & 0 & \bar{\Theta}_{214} & \bar{\Theta}_{215} & \bar{\Theta}_{216} \\ \bar{\Theta}_{221} & 0 & 0 & 0 & \bar{\Theta}_{225} & 0 \\ 0 & \bar{\Theta}_{232} & 0 & 0 & \bar{\Theta}_{235} & 0 \\ \bar{\Theta}_{241} & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\Theta}_{252} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\Theta}_{31} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\Theta}_{322} & 0 & 0 & \bar{\Theta}_{335} & 0 \\ 0 & 0 & 0 & \bar{\Theta}_{344} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\Theta}_{211} = \begin{bmatrix} W_1 A + \bar{\xi}_0 W_2 B_c (M_1 - I)C + \bar{\alpha}_0 W_2 B_c C & W_1 B C_c + W_2 A_c \\ W_3 A + \bar{\xi}_0 W_2 B_c (M_1 - I)C + \bar{\alpha}_0 W_2 B_c C & W_3 B C_c + W_2 A_c \end{bmatrix}$$

and the other matrices in $\bar{\Theta}_{21}$, $\bar{\Theta}_{22}$, $\bar{\Theta}_{31}$ and $\bar{\Theta}_{33}$ are the corresponding matrices in $\bar{\Psi}_{21}$, $\bar{\Psi}_{22}$, $\bar{\Psi}_{31}$ and $\bar{\Psi}_{33}$ with $\tilde{B}_c^{(i)}$ replaced by $W_2\tilde{B}_c^{(i)}(i = 0,1,\cdots,d)$.

We use $\mathrm{T} \in \mathbb{R}^{n\times n}$ to denote the corresponding invertible matrix, *i.e.*, $\mathrm{T}B = \begin{bmatrix} I_{p\times p} \\ 0_{(n-p)\times p} \end{bmatrix}$. For each B of full column rank, the corresponding T is not unique, such as a special form of

$$\mathrm{T} = \begin{bmatrix} (B^T B)^{-1} B^T \\ B^{\perp} \end{bmatrix}$$

where $B^{\perp}$ represents an orthogonal basis for the null space of B [34]. In addition, we define a constant matrix $U_{(n-p)\times p}$ as

$$U_{(n-p)\times p} = \begin{cases} \begin{bmatrix} I_{(n-p)\times(n-p)} & 0_{(n-p)\times(2p-n)} \end{bmatrix} & n \le 2p \\ \begin{bmatrix} I_{p\times p} \\ 0_{(n-2p)\times p} \end{bmatrix} & n > 2p \end{cases}$$

Now, we define W_1 and W_3 as

$$W_1 = \begin{bmatrix} W_{11} & W_{12} \\ \varepsilon_1 U_{(n-p)\times p} W_{11} & W_{22} \end{bmatrix} T, \; W_3 = \begin{bmatrix} \varepsilon_2 W_{11} & W_{32} \\ \varepsilon_3 U_{(n-p)\times p} W_{11} & W_{42} \end{bmatrix} T$$

We can obtain (8.27) directly by defining the new variables $\tilde{A}_c = W_2 A_c$, $\tilde{B}_c = W_2 B_c$ and $\tilde{C}_c = W_{11} C_c$. Because W_1 and W_2 are all nonsingular, the inverse of W_{11} and W_2 exist. Then, (8.26) holds. The proof is completed.

Remark 8.8 It is worth mentioning that the design strategy presented in this chapter can easily be extended to the cases that there are multiple stochastic nonlinearities in the system or in other words, there are multiple stochastic disturbances in the system, and the designed controller is transmitted through networks where the data losses will also happen.

Furthermore, the minimal attenuation level γ can be obtained by solving the following problem.

Problem 8.1 The problem of H_∞ controller design is equivalent to the following optimization problem

$$\min_{\substack{P_i>0, S_i>0, H_i>0 (i=1,2,3), \\ \lambda_1>0, \lambda_2>0, Q_{j1}>0, Q_{j2}>0, \\ Q_{j3}>0 (j=1,2,\cdots,d),, \tilde{A}_c, \tilde{B}_c, \tilde{C}_c}} \varpi$$

subject to (8.25) where $\varpi = \gamma^2$

The corresponding optimal H_∞ performance level γ can be obtained by $\gamma = \sqrt{\varpi}$.

Remark 8.9 Up to now, the controller design algorithm has been developed based on LMI technique for a class of stochastic nonlinear systems with randomly occurring sensor saturations and transmission delays in form of (8.1). From [24], this LMI-based algorithm has a polynomial-time complexity, which is bounded by $(\kappa N^3 \log(c/\varepsilon))$, where κ is the total row size of the LMI system, N is the total number of scalar decision variables, c is a data-dependent scaling factor, and ε is relative accuracy set for algorithm. Note that the dimensions of the system variables x_k, y_k and z_k are n, r and m, respectively. As for the proposed algorithm in this chapter, from Theorem 8.2, we have $= 3n^2d + 7n^2 + p^2 + nd + 3n - p - np + 4$, $\kappa = 9n + 5nd + (d + 1)(r + 1) + m + l + q$. Therefore, the time complexity of our algorithm is $O(n^7d^4)$ which is obviously dependent on the maximal delay d and the dimension n of the state variable.

8.4 Simulation example

In this section, an example for a quadrotor aircraft model is given to demonstrate the effectiveness and applicability of the proposed controller design approach. For the system in [35], with a fixed yaw angle r (0.095 rad), the state-space model is shown as follows [36]:

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & \frac{r(I_y - I_z)}{I_x} & 0 & 0 \\ \frac{r(I_z - I_x)}{I_y} & 0 & 0 & 0 \\ 1 & 0 & 0 & r \\ 0 & 1 & -r & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ \phi \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{L}{I_x} & 0 \\ 0 & \frac{L}{I_y} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{roll} \\ \delta_{pitch} \end{bmatrix}$$

Here, p and q are body angular rates in body frame; ϕ and θ are roll and pitch Euler angles, respectively. $L = 0.3085\, m$ is the distance from the motor to the helicopter's center of mass; $I_x = 0.3428\, kg\, m^2$, $I_y = 0.4112\, kg\, m^2$, $I_z = 0.7863\, kg\, m^2$ are the body moments of inertia with respect to the x, y and z axes, respectively.

A discrete-time system is obtained with a sampling period $T = 0.01\, s$. The matrices of the model are given as follows:

$$A = \begin{bmatrix} 0 & -0.0951 & -0.0002 & -0.0001 \\ 0.0952 & 0 & -0.0005 & -0.0004 \\ 1 & 0 & 0 & 0.0950 \\ 0 & 1 & -0.0950 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.9813 & 0 & 1.2958 & 0.2802 \\ 0 & -1.4322 & 0 & -0.4130 \\ 0 & 0 & 1.5462 & 0 \\ -0.0155 & 0.3756 & 0 & -1.3324 \end{bmatrix}, \quad D = \begin{bmatrix} 0.0481 \\ 0.0568 \\ 0.0468 \\ 0.0500 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0090 & 0 \\ 0 & 0.0075 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \underbrace{[1 \quad 1 \quad \cdots \quad 1]}_{(d+1)r}^{T}.$$

The nonlinear function $f(x) = [f_1(x) \quad f_2(x) \quad f_3(x) \quad f_4(x)]^T$ is given by

$$\begin{cases} f_1(x) = \tanh(-x_1) + 0.2x_1 + 0.1x_3 \\ f_2(x) = \tanh(-x_2) + 0.3x_2 + 0.1x_4 \\ f_3(x) = 0.1x_1 + 0.2x_3 - \tanh(x_3) \\ f_4(x) = 0.1x_1 + 0.3x_2 - \tanh(x_4) + 0.2x_4 \end{cases}$$

It can be easily verified that

$$R_1 = \begin{bmatrix} -0.8 & 0 & 0.1 & 0 \\ 0 & -0.7 & 0 & 0.1 \\ 0.1 & 0 & -0.8 & 0 \\ 0.1 & 0.3 & 0 & -0.8 \end{bmatrix}, R_2 = \begin{bmatrix} 0.2 & 0 & 0.1 & 0 \\ 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0 & 0.2 & 0 \\ 0.1 & 0.3 & 0 & 0.2 \end{bmatrix}$$

And the sensor nonlinearity is taken as

$$\phi(y_k) = \frac{M_1 + M_2}{2} y_k + \frac{M_2 - M_1}{2} \sin(y_k)$$

where $M_1 = diag\{0.4 \quad 0.4 \quad 0.5 \quad 0.6\}$, $M_2 = diag\{0.5 \quad 0.6 \quad 0.6 \quad 0.7\}$. For simulation purpose, we assume that the maximum transmission delay is $d = 2$, the randomly varying time delay is $\tau_k = 2 + \frac{1+(-1)^k}{2}$, the mathematical expectations of the channel coefficients are selected as $\bar{\zeta}_0 = 0.4$, $\bar{\zeta}_1 = 0.5$, $\bar{\zeta}_2 = 0.3$, $\bar{\delta} = 0.4$ and $\bar{\beta} = 0.6$. We assume the initial conditions of the system to be $x_0 = \hat{x}_0 = [0.1 \quad -0.26 \quad 0 \quad 0.03]^T$ and the disturbance inputs are assumed to be $w_k = 2e^{0.1k}\sin(0.1\pi k)$ and $v_k = 0.5rand(1)/(1 + 0.05k)$, respectively. Let the H_∞ performance $\gamma = 0.5$ with the above parameters and by using matlab LMI toolbox to solve the feasibility of LMI in (8.24), we can obtain the controller parameters as

$$A_c = \begin{bmatrix} 0.0256 & 0.0019 & -0.0113 & 0.0025 \\ 0.0698 & 0.0421 & 0.0040 & 0.0040 \\ 0.1717 & 0.0500 & -0.0039 & 0.0345 \\ 0.1639 & 0.2027 & 0.0075 & 0.0288 \end{bmatrix},$$

$$B_c^{(0)} = \begin{bmatrix} 0.0186 & 0.0072 & -0.0079 & 0.0034 \\ 0.0291 & 0.0178 & -0.0184 & -0.0008 \\ 0.0776 & 0.0190 & -0.0576 & 0.0255 \\ 0.0161 & 0.0849 & 0.0004 & -0.0225 \end{bmatrix},$$

$$B_c^{(1)} = \begin{bmatrix} -0.0046 & -0.0016 & 0.0019 & -0.0008 \\ -0.0071 & -0.0041 & 0.0044 & 0.0002 \\ -0.0192 & -0.0044 & 0.0139 & -0.0059 \\ -0.0039 & -0.0204 & -0.0002 & 0.0054 \end{bmatrix},$$

$$B_c^{(2)} = \begin{bmatrix} -0.0139 & -0.0056 & 0.0060 & -0.0025 \\ -0.0218 & -0.0135 & 0.0139 & 0.0006 \\ -0.0582 & -0.0142 & 0.0434 & -0.0191 \\ -0.0121 & -0.0633 & -0.0003 & 0.0170 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -0.0007 & 0.0004 & 0.0007 & -0.0001 \\ -0.0037 & -0.0021 & -0.0003 & -0.0003 \end{bmatrix}$$

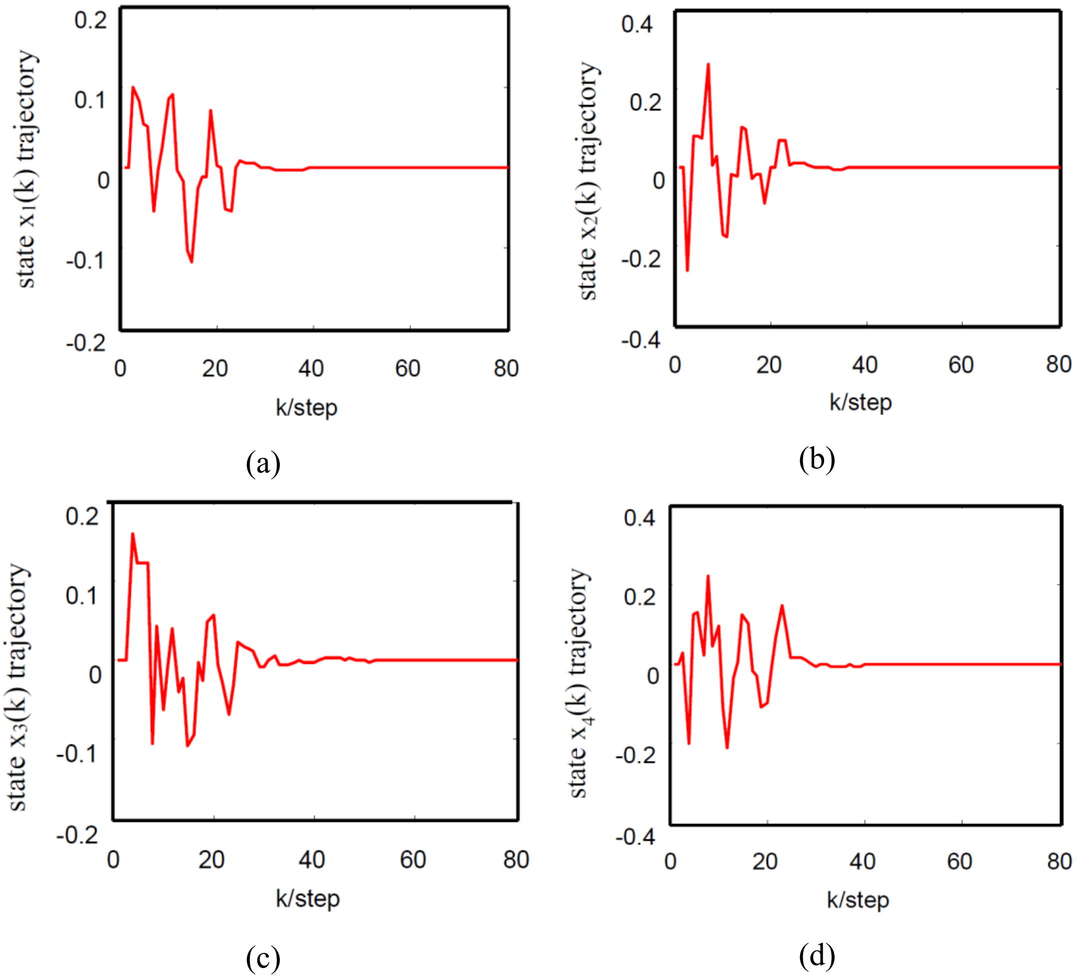

Figure 8.2 State response of the closed-loop system. (a) The first state response; (b) The second state response; (c) The third state response; (d) The fourth state response.

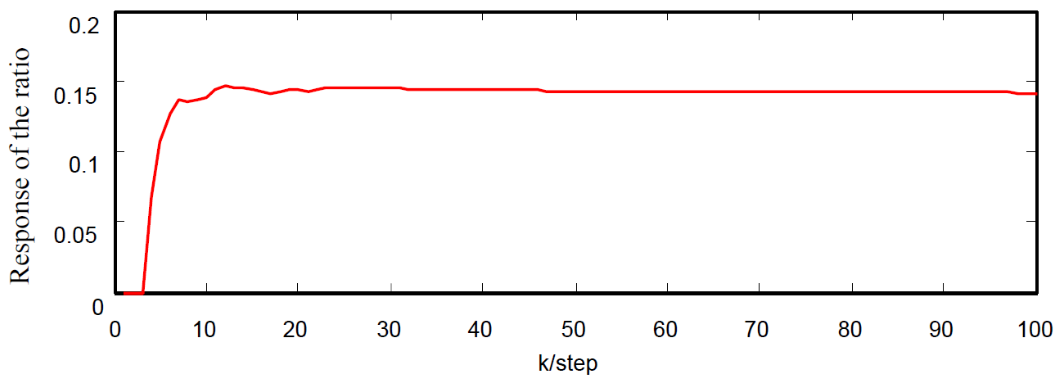

Figure 8.3 Response of the ratio.

Figure 8.2 gives the closed-loop state response. Figure 8.3 shows the response of the ratio $\varphi = \sqrt{\sum_{k=0}^{\infty}\|z_k\|^2}/\sqrt{\sum_{k=0}^{\infty}\|\widetilde{w}_k\|^2}$ of the closed-loop system when the initial conditions are assumed to be zero. From Figure 8.2 and Figure 8.3, we observe that the system is stable with guaranteed H_∞ performance.

Now, let us take a look at the effects on the system performance of the packet loss rates and the time delay rates. We give the comparisons for different probabilities of $\bar{\zeta}_0$, $\bar{\zeta}_1$ and $\bar{\zeta}_2$ changed from 0.2 to 1.0 under the conditions of $\bar{\delta} = 0.4$ and $\bar{\beta} = 0.2$. Table 8.1 shows the simulation results where $P_{1-delay}$ denotes the probability of possible one-step delay which can be calculated by $(1-\bar{\zeta}_0)\bar{\zeta}_1$, $P_{2-delay}$ denotes the probability of possible two-step delay which can be calculated by $(1-\bar{\zeta}_0)(1-\bar{\zeta}_1)\bar{\zeta}_2$, and P_{drop} denotes probability of the possible packet dropouts which can be calculated by $1-\bar{\zeta}_0-(1-\bar{\zeta}_0)\bar{\zeta}_1-(1-\bar{\zeta}_0)(1-\bar{\zeta}_1)\bar{\zeta}_2$ with the on-time rate being $\bar{\zeta}_0$ for a packet.

Table 8.1 Comparison of the minimum H_∞ performance for different time delay rates and packet dropout rates.

$\bar{\zeta}_0/\bar{\zeta}_1/\bar{\zeta}_2$	$P_{1-delay}$	$P_{2-delay}$	P_{drop}	γ_{min}
0.2/0.2/0.2	0.16	0.128	0.512	0.0259
0.4/0.4/0.4	0.24	0.144	0.216	0.0258
0.6/0.6/0.6	0.24	0.096	0.064	0.0254
0.8/0.8/0.8	0.16	0.032	0.008	0.0247
1.0/1.0/1.0	0	0	0	0.0236

It can be seen from Table 8.1 that the H_∞ performance is degraded when the rate of a packet received on time is becoming smaller and the packet dropout rate becoming larger. Furthermore, we give a closer look at the effects on the system performance of stochastic nonlinearity and sensor saturation /nonlinearity.

The probabilities of $\bar{\zeta}_0$, $\bar{\zeta}_1$ and $\bar{\zeta}_2$ are unchanged, *i.e.*, $\bar{\zeta}_0 = 0.4$, $\bar{\zeta}_1 = 0.5$, $\bar{\zeta}_2 = 0.3$, and the probability of $\bar{\beta}$ is changed from 0.1 to 0.9 with $\bar{\delta} = 0.6$, while the probability of $\bar{\delta}$ is changed from 0.1 to 0.9 with $\bar{\beta} = 0.6$. Table 8.2 gives the simulation results.

Table 8.2 Comparison of the minimum H_∞ performance for different stochastic nonlinearity rates and sensor saturation rates.

$\bar{\beta}$	0.1	0.3	0.5	0.7	0.9
γ_{min}	0.0257	0.0258	0.0266	0.0270	0.0289
$\bar{\delta}$	0.1	0.3	0.5	0.7	0.9
γ_{min}	0.0263	0.0271	0.0271	0.0283	0.0283

It is obvious from Table 8.2 that when the nonlinearity is becoming serious, the H_∞ performance of the system is going to bad. However, the effect of stochastic nonlinearities on the system performance is bigger than the sensor saturation/nonlinearity. It is mainly because the sensor saturation is dealt with the decomposition of a linear part and a nonlinear part which can weaken the effect of nonlinearity.

8.5 Conclusions

We have investigated the H_∞ controller design problem for a class of discrete-time network-based nonlinear systems with random bounded transmission delays, consecutive packet losses and randomly occurring sensor saturations in this chapter. The randomness is described by Bernoulli distributed variables and the nonlinearities both in the state and the sensor output are described by sector-bounded functions. An observer-based nonlinear H_∞ controller is designed such that the closed-loop system is asymptotically stable in mean square sense with a prescribed H_∞ performance. The desired parameters of the proposed controller are given in terms of LMI. Finally, a quadrotor aircraft system is given to show the feasibility and applicability of the proposed controller design scheme. Our future research topic is to develop the H_∞ controller design techniques for a class of T-S fuzzy systems subject to randomly occurring sensor saturations under the network circumstances by taking the network-induced random phenomena into consideration.

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Appendix A About H_∞ Control Theory

Since its birth in the late 1950s, modern control theory has developed rapidly, and been successfully applied in the aerospace field in the 1960s. LQG design (referred to H_2 control) based on Kalman-Bucy filtering and the optimal control theory has emerged. Many results of modern control theory are beautiful in theory, but they have been unsuccessfully applied in industry. This is mainly attributed to the neglect of the uncertainty of the object and exact requirements of interfering signals of the system. The robust control theory makes up for the shortcomings of modern control theory that requires accurate mathematical models of objects. It considers uncertainties including the parameters of system model and external disturbances, resulting in that the analysis and synthesis methods of the system are more effective and practical.

H_∞ control is a branch of robust control. It uses the optimal sensitivity, which means the interference has the minimum influence on the output, as the basic formulation. A well-known paper published by Canadian scholar G. Zames in 1981 (G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control* **1981**, *26*, 301-302) can be seen as a pioneer of modern robust control, especially H_∞ control. Zames assumed that the interfering signal belongs to a finite energy signal set, and then proposed to use the norm of its corresponding sensitivity function as an indicator. The design goal is to minimize the H_∞ norm of the error under the condition of the worst possible interference. Then, the interference problem is turned into an output feedback control one that how to stabilize the closed-loop system and achieve the minimum value of corresponding H_∞ norm index.

A basic feedback control system is shown in Figure A.1.

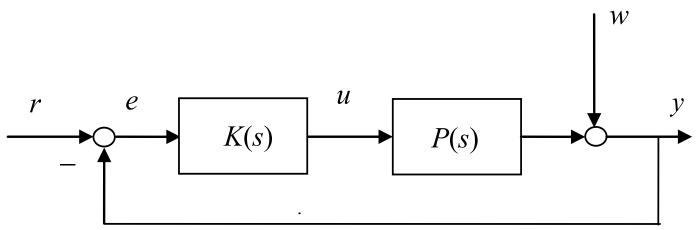

Figure A.1 Block diagram of the basic feedback control system.

where $K(s)$ is the controller, w is the interference signal, r is the reference input,

u is the control input, e is the control error signal, and y is the output signal. The frequency characteristics of open-loop and closed-loop of the system are:

$$G_K(j\omega) = P(j\omega)K(j\omega), G_B = \frac{P(j\omega)K(j\omega)}{1 + P(j\omega)K(j\omega)}$$

If $P(s)$ has an error $P(s) = P_0(s) + \Delta P(s)$, the corresponding frequency characteristics of open-loop and closed-loop also have errors.

$$\Delta G_K(j\omega) = G_K(j\omega) - G_{K0}(j\omega),\ \Delta G_B(j\omega) = G_B(j\omega) - G_{B0}(j\omega)$$

where

$$G_{K0}(j\omega) = P_0(j\omega)K(j\omega), G_{B0} = \frac{P_0(j\omega)K(j\omega)}{1 + P_0(j\omega)K(j\omega)}$$

are nominal functions of frequency characteristics of open-loop and closed-loop, respectively. Following can be obtained through simple derivation:

$$\frac{\Delta G_B(j\omega)}{G_B(j\omega)} = \frac{1}{1 + P_0(j\omega)K(j\omega)} \frac{\Delta G_K(j\omega)}{G_K(j\omega)}$$

The transfer function $S(s) = \frac{1}{1+P_0(s)K(s)}$ embodies the gain from the relative deviation of the open-loop frequency characteristics $\frac{\Delta G_K}{G_K}$ to the closed-loop frequency characteristic $\frac{\Delta G_B}{G_B}$. Thus, if we design the controller K with a small enough value of gain S, *i.e.*,

$$|S(j\omega)| < \varepsilon,\ \varepsilon \text{ is a small enough value}$$

the deviation of the closed-loop frequency characteristics will be suppressed within the region allowed by the project. The transfer function $S(s)$ is called the sensitivity function of the system. It is equal to the closed-loop transfer function from the interference w to the output. Thus, a decrease in gain $S(s)$ is equivalent to reducing the effect of interference on the control error. The following definition is introduced:

$$\|S(s)\|_\infty = \sup_{\omega \in R} \bar{\sigma}[S(j\omega)]$$

where $\bar{\sigma}(\cdot)$ is the maximum singular value, $\bar{\sigma}(A) = \{\lambda_{\max}(A^* A)\}^{\frac{1}{2}}$, A^* is the conjugate transpose matrix of A.

H_∞ control problem is how to design a controller K that can make the closed loop system stable and satisfy $\|S(s)\|_\infty < \varepsilon$ with given $\varepsilon > 0$.

For H_∞ theory, the interference signal is uncertain, and belongs to a descriptive set

$$L_2 = \left\{ w(t) | \int_0^\infty w^2(t)dt < \infty \right\}$$

L_2 contains signals with limited energy. The suppressing interference $w \in L_2$ affects the system performance, so a scalar γ indicating the level of interference suppression is introduced. The controller K is designed to satisfy

$$\|z\|_2^2 < \gamma^2 \|w\|_2^2, \forall w \in L_2 \tag{A.1}$$

where z is the output signal. The following definition is introduced

$$\|T_{zw}(j\omega)\|_{\infty} = \sup_{w \neq 0} \frac{\|z\|_2}{\|w\|_2}$$

where $T_{zw}(s)$ is the closed-loop transfer function from w to z. Then, Equation (A.1) is equivalent to the problem how to design the controller K with the minimum value of γ, which is the optimal design problem of H_{∞}.

Appendix B About Linear Matrix Inequality

In the early 1990s, with the advent of interior point method for solving convex optimization problems, the linear matrix inequality (LMI) method had attracted attention from the control community. In 1995, Matlab introduced the LMI toolbox, resulting in that the LMI attracted more and more attention. Many control problems can be transformed into problems of the feasibility of an LMI or convex optimization problems with LMI constraints. Currently, LMI has become an important tool for analysis and synthesis of robust control of uncertain systems.

The general representation of LMI is given as follows:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i < 0 \tag{A.2}$$

where $x_i \ (i = 1, \cdots, m)$ is m real variables, called decision variable of Equation (A.2); $x = (x_1, \cdots, x_m)^T \in R$ is a decision vector composed of decision variables. $F_i = F_i^T \in R^{n\times n}$, $i = 0,1, \cdots, m$ is a given set of real symmetric matrices.

Many problems in system and control do not seem to be LMI problems at first, or do not have the form of Equation (A.2). However, they can be converted into LMI problems with form of Equation (A.2) by appropriate processing. Schur complement is often used to convert nonlinear inequalities into equivalent LMI, which greatly facilitates the solution of the problem.

Lemma A.1 (Schur complement) Given a symmetric matrix $S \in R^{n\times n}$, which is represented by blocks

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

where $S_{11} \in R^{r\times r}$, $S_{12} \in R^{r\times(n-r)}$, $S_{21} \in R^{(n-r)\times r}$, $S_{22} \in R^{(n-r)\times(n-r)}$, the following statements are equivalent:

$$\begin{aligned} &(1)\ S < 0; \\ &(2)\ S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0; \\ &(3)\ S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0. \end{aligned} \tag{A.3}$$

Quadratic matrix inequalities are often encountered in control theory research, and can be converted into LMI by Lemma A.1. This is one of the main reasons why LMI can be widely used in control theory research.

The LMI toolbox provides an LMI solver for solving the following three problems, where x represents the decision variable vector, F and G are symmetric affine functions of matrix value, c is a given constant vector.

(1) **LMI Problem (LMIP)** The problem is looking for x that satisfies the LMI

$$F(x) < 0$$

The corresponding solver is feasp.

(2) **Eigenvalue Problem (EVP)** A minimization problem for a linear objective function with LMI constraints:

$$min\ \ c^T x$$
$$\text{s.t. } F(x) < 0$$

The corresponding solver is mincx.

(3) **Generalized Eigenvalue Problem (GEVP)** Under the constraint of a matrix inequality, the minimization problem of the greatest generalized eigenvalues of two affine matrix functions

$$min\ \ \lambda$$
$$\text{s.t. } G - \lambda F < 0$$

The corresponding solver is gevp.

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